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Trace au bord de solutions d'équations de Hamilton-Jacobi elliptiques et trace initiale de solutions d'équations de la chaleur avec absorption sur-linéaire

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A mes parents...

Trace au bord de solutions d'équations de Hamilton-Jacobi elliptiques et trace initiale de solutions d'équations de la chaleur avec absorption sur-linéaire

Résumé

Cette thèse est constituée de trois parties.

Dans la première partie, on s'intéresse au problème de *trace au bord* d'une solution positive de l'équation de Hamilton-Jacobi (E1) $-\Delta u + g(|\nabla u|) = 0$ dans un domaine borné Ω de $\mathbb{R}^N$, satisfaisant (E2) $u = \mu$ sur $\partial\Omega$. Si $g(r) \geq r^q$ avec $q > 1$, on prouve que toute solution positive de (E1) admet une trace au bord considérée comme une mesure de Borel régulière, pas nécessairement localement bornée. Si $g(r) = r^q$ avec $1 < q < q_c = \frac{N+1}{N}$, on montre l'existence d'une solution positive dont la trace au bord est une mesure de Borel régulière $\nu \not\equiv \infty$ et on caractérise les singularités frontières isolées de solutions positives. Si $g(r) = r^q$ avec $q_c \leq q < 2$, on établit une condition nécessaire de résolution en terme de capacité de Bessel $C_{\frac{2-q}{q},q'}$. On étudie aussi des ensembles éliminables au bord pour des solutions modérées et sigma-modérées.

La deuxième partie est consacrée à étudier la limite, lorsque $k \to \infty$, de solutions d'équation $\partial_t u - \Delta u + f(u) = 0$ dans $\mathbb{R}^N \times (0,\infty)$ avec donnée initiale $k\delta_0$ où δ_0 est la masse de Dirac concentrée à l'origine et f est une fonction positive, continue, croissante et satisfait $f(0) = f^{-1}(0) = 0$. On prouve, sous certaines hypothèses portant sur f, qu'il existe essentiellement trois types de comportement possible en fonction des valeurs finies ou infinies des intégrales $\int_1^\infty f^{-1}(s)ds$ et $\int_1^\infty F^{-1/2}(s)ds$, où $F(s) = \int_0^s f(r)dr$. Grâce à ces résultats, on donne une nouvelle construction de la trace initiale et quelques résultats d'unicité et de non-unicité de solutions dont la donnée initiale n'est pas bornée.

Dans la troisième partie, on élargit le cadre de nos investigations et généralise les résultats obtenus dans la deuxième partie au cas où l'opérateur est non-linéaire. En particulier, on s'intéresse à des propriétés qualitatives de solutions positives de l'équation $\partial_t u - \Delta_p u + f(u) = 0$ dans $\mathbb{R}^N \times (0,\infty)$ où $p > 1$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ et f est une fonction continue, croissante, positive et satisfait $f(0) = 0 = f^{-1}(0)$. Si $p > \frac{2N}{N+1}$, on fournit une condition suffisante portant sur f pour l'existence et l'unicité des solutions fondamentales de données initiales $k\delta_0$ et on étudie la limite, lorsque $k \to \infty$, qui dépend du fait que f^{-1} et $F^{-1/p}$ soient intégrables à l'infini ou pas, où $F(s) = \int_0^s f(r)dr$. On donne aussi de nouveaux résultats de non-unicité de solutions avec donnée initiale non bornée. Si $p \geq 2$, on prouve que toute solution positive admet une trace initiale dans la classe de mesures de Borel régulières positives. Finalement on applique les résultats ci-dessus au cas modèle $f(u) = u^\alpha \ln^\beta(u+1)$ avec $\alpha > 0$ et $\beta > 0$.

Mots clés : équations elliptiques quasilinéaires, singularités isolées, mesures de Radon, mesures de Borel, capacités de Bessel, trace au bord, singularités éliminables, absorption faiblement sur-linéaire, trace initiale, condition de Keller-Osserman, équations de la chaleur dégénérées.

Boundary trace of solutions to elliptic Hamilton-Jacobi equations and initial trace of solutions to heat equations with superlinear absorption

Abstract

This thesis is divided into three parts.

In the first part, we study the boundary value problem with measures for the Hamilton-Jacobi equation (E1) $-\Delta u + g(|\nabla u|) = 0$ in a bounded domain Ω in $\mathbb{R}^N$, satisfying (E2) $u = \mu$ on $\partial\Omega$ and provide a condition on g for which the problem (E1)-(E2) can be solved with any positive bounded measure. When $g(r) \geq r^q$ with $q > 1$, we prove that any positive solution of (E1) admits a boundary trace which is an outer regular Borel measure, not necessarily bounded. When $g(r) = r^q$ with $1 < q < q_c = \frac{N+1}{N}$, we prove the existence of a positive solution with a general outer regular Borel measure $\nu \not\equiv \infty$ as boundary trace and we characterize the boundary isolated singularities of positive solutions. When $g(r) = r^q$ with $q_c \leq q < 2$, we show that a necessary condition for solvability is that μ must be absolutely continuous with respect to the Bessel capacity $C_{\frac{2-q}{q},q'}$. We also characterize boundary removable sets for moderate and sigma-moderate solutions.

The second part is devoted to investigate the limit, when $k \to \infty$, of the solutions of $\partial_t u - \Delta u + f(u) = 0$ in $\mathbb{R}^N \times (0,\infty)$ with initial data $k\delta_0$, where δ_0 is the Dirac mass concentrated at the origin and f is a nonnegative, continuous, nondecreasing function satisfying $f(0) = f^{-1}(0) = 0$. We prove that there exist essentially three types of possible behaviour according f^{-1} and $F^{-1/2}$ belong or not to $L^1(1,\infty)$, where $F(s) = \int_0^s f(r)dr$. We use these results for providing a new construction of the initial trace and some uniqueness and non-uniqueness results for solutions with unbounded initial data.

The main goal of the third part is to investigate the initial value problem with unbounded nonnegative functions or measures for the equation $\partial_t u - \Delta_p u + f(u) = 0$ in $\mathbb{R}^N \times (0,\infty)$ where $p > 1$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ and f is a continuous, nondecreasing nonnegative function such that $f(0) = f^{-1}(0) = 0$ and to extend the results obtained in the second part to the cas $p \neq 2$. If $p > \frac{2N}{N+1}$, we provide a sufficient condition on f for existence and uniqueness of the fundamental solutions satisfying the initial data $k\delta_0$ and we study their limit, when $k \to \infty$, according f^{-1} and $F^{-1/p}$ are integrable or not at infinity, where $F(s) = \int_0^s f(r)dr$. We also give new results dealing with non uniqueness for the initial value problem with unbounded initial data. If $p \geq 2$, we prove that any positive solution admits an initial trace in the class of positive Borel measures. As a model case we consider the case $f(u) = u^\alpha \ln^\beta(u+1)$ with $\alpha > 0$ and $\beta > 0$.

Key words : quasilinear elliptic equations, isolated singularities, Radon measures, Borel measures, Bessel capacities, boundary trace, removable singularities, weakly superlinear absorption, initial trace, Keller-Osserman condition, degenerate heat equations.

Notations

Divers :

Ω : domaine de $\mathbb{R}^N$.

$Q_T^\Omega = \Omega \times (0,T)$, $Q_T = \mathbb{R}^N \times (0,T)$, $Q_\infty = \mathbb{R}^N \times (0,\infty)$.

$B_r(x_0)$: la boule de centre $x_0 \in \mathbb{R}^N$ et de rayon r. Pour simplifier, B_r désigne la boule de centre à l'orgine et de rayon r.

S^{N-1} : la sphère unité de $\mathbb{R}^N$.

dS : élément de volume sur $\partial\Omega$.

$d(x) = \operatorname{dist}(x, \partial\Omega)$.

Espaces de Lebesgue :

$L^p(\Omega) = \{u : \Omega \to \mathbb{R}$ mesurable $: \int_\Omega |u|^p < \infty\}$ pour tout $p \in [1,\infty)$.

Espaces de Sobolev :

$W^{k,p}(\Omega) = \{u : \Omega \to \mathbb{R}$ mesurable $: D^\alpha u \in L^p(\Omega)$ pour tout $\alpha \in \mathbb{N}^N$ t. q. $|\alpha| \leq k\}$ pour tout $k \in \mathbb{N}^*$ et $p \in [1,\infty)$.

Espaces de mesures : Soit $\alpha \in [0,1]$.

$\mathfrak{M}(\Omega)$ (resp. $\mathfrak{M}(\partial\Omega)$) : espace de mesures de Radon sur Ω (resp. $\partial\Omega$).

$\mathfrak{M}_+(\Omega)$ (resp. $\mathfrak{M}_+(\partial\Omega)$) : espace de mesures de Radon positives sur Ω (resp. $\partial\Omega$).

$\mathfrak{M}^b(\Omega)$ (resp. $\mathfrak{M}^b(\partial\Omega)$) : espace de mesures de Radon bornées sur Ω (resp. $\partial\Omega$).

$\mathfrak{M}^b_+(\Omega)$ (resp. $\mathfrak{M}^b_+(\partial\Omega)$) : espace de mesures de Radon bornées positives sur Ω (resp. $\partial\Omega$).

$\mathfrak{M}_{d^\alpha}(\Omega) = \{\mu \in \mathfrak{M}(\Omega) : \int_\Omega d^\alpha d\,|\mu| < \infty\}$. If $\alpha = 0$, $\mathfrak{M}_{d^0}(\Omega) = \mathfrak{M}^b(\Omega)$.

$\mathcal{B}^{reg}_+(\Omega)$ (resp. $\mathcal{B}^{reg}_+(\partial\Omega)$) : espace de mesures de Borel régulières sur Ω (resp. $\partial\Omega$).

Espaces de Marcinkiewicz (ou espaces de Lebesgue faibles) : Soient $p > 1$, $\alpha \in [0,1]$ et μ une mesure de Borel positive sur Ω.

$M^p(\Omega; d\mu) = \{u \in L^1_{loc}(\Omega; d\mu) : \|u\|_{M^p(\Omega;d\mu)} < \infty\}$ où

$$\|u\|_{M^p(\Omega;d\mu)} = \inf\left\{c \in [0,\infty) : \int_E |u|\,d\mu \leq c\left(\int_E d\mu\right)^{1-\frac{1}{p}} \quad \forall E \subset \Omega, E \text{ Borel}\right\}.$$

Si $d\mu = dx$, on utilisera la notion $M^p(\Omega)$ au lieu de $M^p(\Omega; dx)$.

Si $d\mu = d^\alpha dx$, on utilisera la notion $M^p_{d^\alpha}(\Omega)$ au lieu de $M^p(\Omega; d^\alpha dx)$.

Espaces potentiels de Bessel : Soient $\alpha \in \mathbb{R}$ et $p > 1$.

$G_\alpha = \mathcal{F}^{-1}((1 + |\xi|^2)^{-\frac{\alpha}{2}})$: noyau de Bessel d'ordre α, où $\mathcal{F}^{-1}$ est la transformation de Fourier inverse sur l'espace de Schwartz $\mathcal{S}(\mathbb{R}^N)$.

$L^{\alpha,p}(\mathbb{R}^N) = \{f : f = G_\alpha * g, g \in L^p(\mathbb{R}^N)\}$: espace potentiel de Bessel d'ordre α et de degré p avec la norme

$$\|f\|_{L^{\alpha,p}(\mathbb{R}^N)} = \|g\|_{L^p(\mathbb{R}^N)} = \|G_{-\alpha} * f\|_{L^p(\mathbb{R}^N)}.$$

Si $\alpha \in \mathbb{N}^*$ et $1 < p < \infty$, $W^{\alpha,p}(\mathbb{R}^N) = L^{\alpha,p}(\mathbb{R}^N)$ et il existe $c > 0$ telle que

$$c^{-1} \|f\|_{L^{\alpha,p}(\mathbb{R}^N)} \leq \|f\|_{W^{\alpha,p}(\mathbb{R}^N)} \leq c \|f\|_{L^{\alpha,p}(\mathbb{R}^N)} \quad \forall f \in W^{\alpha,p}(\mathbb{R}^N).$$

Capacités de Bessel : Soient $\alpha > 0$ et $p > 1$.

$C_{\alpha,p}(K) = \inf\{\|\phi\|_{L^{\alpha,p}(\mathbb{R}^N)} : \phi \in \mathcal{S}(\mathbb{R}^N), \phi \geq 1$ dans un voisinage de K$\}$ si K est compact,

$C_{\alpha,p}(G) = \sup\{C_{\alpha,p}(K) : K \subset G, K$ compact$\}$ si G est ouvert,

$C_{\alpha,p}(E) = \inf\{C_{\alpha,p}(G) : E \subset G, G$ ouvert$\}$ pour tout ensemble E.

Conventions

La numération choisie pour l'énoncé des théorèmes, propositions, lemmes, corollaires,...dans l'introduction est indépendante de celle dans les chapitres 1, 2 et 3.

Les symboles employés dans chaque partie comme ϕ, φ, ψ, J, K, u_ϵ, v_ϵ,...ne sont valables que dans cette partie-là. Par exemple le symbole $u_{k\delta_0}$ dans chapitre 1 est défini de manière différente de celui du chapitre 2. Dans chaque partie (y compris l'introduction), des références citées se trouvent à la fin de cette partie-là.

C_i, c_i et c_i' ($i \in \mathbb{N}$) désignent des constantes dépendantes de données initiales (comme N, p, q...) et d'autres quantités données (comme des fonctions tests, des fonctions propres, des valeurs propres,...). Leur valeur peut varier d'une ligne à l'autre.

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Introduction générale

0.1 Equations de Hamilton-Jacobi elliptiques

0.1.1 Trace au bord et singularités isolées au bord

Soient Ω un domaine borné de classe C^2 de $\mathbb{R}^N$ ($N \geq 2$) et g une fonction continue croissante de $\mathbb{R}_+$ dans $\mathbb{R}_+$ s'annulant en 0. Dans le premier chapitre, on s'intéresse aux solutions positives d'équation du type

$$-\Delta u + g(|\nabla u|) = 0 \qquad \text{dans } \Omega, \tag{0.1.1}$$

et on se concentrera en particulier au cas

$$-\Delta u + |\nabla u|^q = 0 \qquad \text{dans } \Omega \tag{0.1.2}$$

où $q \in (1,2)$. On considère d'abord le problème de *trace au bord* associé à l'équation (0.1.1)

$$\begin{cases} -\Delta u + g(|\nabla u|) = 0 & \text{dans } \Omega \\ u = \mu & \text{sur } \partial\Omega \end{cases} \tag{0.1.3}$$

où μ est une mesure sur $\partial\Omega$. Une fonction u est dite solution du problème (0.1.3) si $u \in L^1(\Omega)$, $g(|\nabla u|) \in L^1_d(\Omega)$ avec $d = d(x) := \operatorname{dist}(x, \partial\Omega)$ et si elle satisfait

$$\int_\Omega (-u\Delta\zeta + g(|\nabla u|)\zeta)\, dx = -\int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\mu \tag{0.1.4}$$

pour tout $\zeta \in X(\Omega) := \{\phi \in C^1_0(\overline{\Omega}) : \Delta\phi \in L^\infty(\Omega)\}$, où $\mathbf{n}$ désigne le vecteur normal unitaire sortant de $\partial\Omega$. Notons que l'hypothèse $g(|\nabla u|) \in L^1_d(\Omega)$ est nécessaire afin que $g(|\nabla u|)\zeta$ soit intégrable pour tout $\zeta \in X(\Omega)$, d'où (0.1.4) a un sens.

L'étude de l'équation (0.1.1) et du problème (0.1.3) est inspirée par les travaux de Le Gall [27], [28], de Gmira et Véron [19] et de Marcus et Véron [35]–[42] sur l'équation semilinéaire

$$-\Delta u + h(u) = 0 \qquad \text{dans } \Omega \tag{0.1.5}$$

et le problème de Dirichlet associé avec donnée mesure

$$\begin{cases} -\Delta u + h(u) = 0 & \text{dans } \Omega \\ u = \mu & \text{sur } \partial\Omega, \end{cases} \tag{0.1.6}$$

où h est une fonction continue, croissante de $\mathbb{R}$ dans $\mathbb{R}$ et s'annulant en 0. Gmira et Véron ont mis en évidence l'existence et l'unicité de la solution du problème (0.1.6) sous l'hypothèse

$$\int_1^\infty (h(s) + |h(-s)|) s^{-\frac{2N}{N-1}} ds < \infty \tag{0.1.7}$$

et $\mu \in \mathfrak{M}(\partial\Omega)$. Pour en savoir plus, on renvoie le lecteur aux travaux de Gmira et Véron [19]. Le cas où $h(u) = |u|^{q-1} u$ avec $q > 1$ a été largement étudié par Marcus et Véron [35]–[38], [40], [42] et l'équation

$$-\Delta u + |u|^{q-1} u = 0 \qquad \text{dans } \Omega \tag{0.1.8}$$

est bien comprise. Remarquons que si $h(u) = |u|^{q-1} u$, la condition (0.1.7) est satisfaite si $1 < q < q_s$ où $q_s := \frac{N+1}{N-1}$ est appelé la valeur d'exposant critique de (0.1.8). On cite ci-dessous les résultats importants concernant (0.1.6)-(0.1.8).

- En introduisant une notion de trace au bord, moyen naturel et efficace de décrire des solutions positives de (0.1.8), Marcus et Véron [37] ont montré que toute solution positive de (0.1.8) dans B_R possède une trace définie de façon unique par un couple $(\mathcal{S}, \mu)$ où $\mathcal{S}$ est un sous-ensemble fermé de ∂B_R et $\mu \in \mathfrak{M}_+(\mathcal{R})$ où $\mathcal{R} = \partial B_R \setminus \mathcal{S}$. Après, dans [42], en utilisant une méthode complètement différente, ils ont établi l'existence de trace au bord de solutions positives de (0.1.6) dans le cas où Ω est un domaine dont la frontière est une variété de classe C^2, au sens faible des mesures.

- Si $1 < q < q_s$, les singularités isolées de solutions positives de (0.1.8) peuvent être complètement classifiées. Plus précisément, si $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ est une solution positive de (0.1.8) qui s'annule sur $\partial\Omega \setminus \{0\}$, alors ou bien elle résout le problème (0.1.6) avec $h(u) = |u|^{q-1} u$ et $\mu = k\delta_0$ pour certain $k \geq 0$ (singularité faible), ou bien [42] $u(x) \approx Cd(x) |x|^{-\frac{q+1}{q-1}}$ lorsque $x \to 0$ (singularité forte).

- De plus, dans le cas sous-critique, il est bien connu qu'étant donné un couple $(\mathcal{S}, \mu)$ où $\mathcal{S}$ est un sous-ensemble fermé de $\partial\Omega$ et μ est une mesure de Radon positive sur $\mathcal{R} := \partial\Omega \setminus \mathcal{S}$, il existe une unique solution positive de (0.1.8) avec la trace au bord $(\mathcal{S}, \mu)$.

- Dans le cas sur-critique $q \geq q_s$, *les singularités isolées sont éliminables*, c'est-à-dire si $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ est une solution positive de (0.1.8) qui s'annule sur $\partial\Omega \setminus \{0\}$, alors u est identiquement nulle. Ce résultat a été tout d'abord établi par Gmira et Véron [19], et après étendu par Le Gall [30] pour le cas $q = 2$ à l'aide d'outils probabilistes, par Dynkin et Kuznetsov [15] pour le cas $q \leq 2$ grâce à une combinaison de méthodes probabilistes et analytiques, et par Marcus et Véron [38], [40] pour le cas général $q \geq q_s$ par une méthode analytique. L'outil clé pour résoudre ce problème est la capacité de Bessel $C_{\frac{2}{q},q'}$ en dimension $N-1$. On énumère ici quelques résultats significatifs. La condition nécessaire et suffisante pour laquelle le problème de Dirichlet associé peut être résolu est que μ est absolument continue par rapport à la capacité $C_{\frac{2}{q},q'}$. De plus, si $K \subset \partial\Omega$ est un compact et $u \in C(\overline{\Omega} \setminus K) \cap C^2(\Omega)$ est une solution positive de (0.1.8) qui s'annule sur $\partial\Omega \setminus K$, alors u est identiquement nulle si et seulement si $C_{\frac{2}{q},q'}(K) = 0$. Une caractérisation complète de solutions positives de (0.1.8) a été établie par Mselati [34] si $q = 2$, par Dynkin [14] pour le cas $q_s \leq q \leq 2$, et finalement par Marcus [33] pour le cas $q \geq q_s$. Ils ont prouvé que toute solution positive u de (0.1.8) est *sigma-modérée*, c'est-à-dire qu'il existe une suite croissante de mesures $\mu_n \in \mathfrak{M}_+(\partial\Omega)$ telle que la suite de solutions correspondantes u_{μ_n} du problème de Dirichlet associé avec la donnée frontière μ_n tende vers u.

On présente ensuite les résultats d'existence et d'éliminabilité de singularités au bord de solutions positives de (0.1.1)-(0.1.3), similaires à ceux rappelés ci-dessus pour (0.1.6)–(0.1.8).

Dans tout ce qui suit, $\mathcal{G}_0$ désigne l'ensemble des fonctions localement Lipschitziennes, croissantes, positives de $\mathbb{R}_+$ dans $\mathbb{R}_+$ et s'annulant en 0. La condition suivante portant sur g est appelée la *condition d'intégrale sous-critique*

$$\int_1^\infty g(s)s^{-\frac{2N+1}{N}}ds < \infty. \tag{0.1.9}$$

Notons que si $g(r) \leq r^q$, la condition (0.1.9) est vérifiée si $1 < q < q_c := \frac{N+1}{N}$. Notre résultat principal d'existence et de stabilité est le suivant :

Théorème 1.1 *On suppose que $g \in \mathcal{G}_0$ satisfait (0.1.9). Alors pout tout $\mu \in \mathfrak{M}_+(\partial\Omega)$, il existe une solution maximale $\overline{u}_\mu$ de (0.1.3). De plus, $\overline{u}_\mu \in M^{\frac{N}{N-1}}(\Omega)$ et $|\nabla\overline{u}_\mu| \in M_d^{\frac{N+1}{N}}(\Omega)$. Finalement, si $\{\mu_n\}$ est une suite de mesures bornées dans $\mathfrak{M}_+(\partial\Omega)$ convergeant vers μ au sens faible et $\{u_{\mu_n}\}$ est une suite de solutions de (0.1.3) avec $\mu = \mu_n$, alors il existe une sous-suite $\{u_{\mu_{n_k}}\}$ qui converge vers une solution u_μ de (0.1.3) dans $L^1(\Omega)$ et $\{g(|\nabla u_{\mu_{n_k}}|)\}$ converge vers $g(|\nabla u|)$ dans $L^1_d(\Omega)$.*

Notre techinique, basée sur la méthode de Gmira et Véron [19], est de construire une suite de solutions approchées et de prouver un résultat de compacité faible à l'aide de (0.1.9) et d'estimations dans l'espace de Marcinkiewicz [7], [19], [50], [51]. L'unicité de la solution du problème (0.1.3) est encore une question ouverte, sauf dans le cas où $h(r) = r^q$ avec $1 < q < q_c$ et $\mu = k\delta_0$ avec $k \geq 0$.

Puisque $\partial\Omega$ est de classe C^2, il existe $\delta^* > 0$ tel que pour tout $\delta \in (0, \delta^*]$ et $x \in \Omega$ vérifiant $d(x) < \delta$, il existe un unique $\sigma(x) \in \partial\Omega$ satisfaisant $|x - \sigma(x)| = d(x)$. On note $\sigma(x) = Proj_{\partial\Omega}(x)$. De plus, si $\mathbf{n} = \mathbf{n}_{\sigma(x)}$ est le vector normal unitaire sortant de $\partial\Omega$ en $\sigma(x)$, on a alors $x = \sigma(x) - d(x)\mathbf{n}_{\sigma(x)}$. Pour tout $\delta \in (0, \delta^*]$, on pose

$$\begin{aligned}
&\Omega_\delta = \{x \in \Omega : d(x) < \delta\},\\
&\Omega'_\delta = \{x \in \Omega : d(x) > \delta\},\\
&\Sigma_\delta = \partial\Omega'_\delta = \{x \in \Omega : d(x) = \delta\},\\
&\Sigma := \Sigma_0 = \partial\Omega.
\end{aligned}$$

Pour tout $\delta \in (0, \delta^*)$, l'application $x \mapsto (d(x), \sigma(x))$ définit un C^1-difféomorphisme de Ω_δ dans $(0, \delta) \times \Sigma$. On peut donc écrire $x = \sigma(x) - d(x)\mathbf{n}_{\sigma(x)}$ pour tout $x \in \Omega_\delta$. Chaque point $x \in \overline{\Omega}_{\delta^*}$ est représenté par un unique couple $(\delta, \sigma) \in [0, \delta^*] \times \Sigma$ sous la forme $x = \sigma - \delta\mathbf{n}_\sigma$. Ce système de coordonnées, appelé *coordonnées de flux*, sert à construire la trace au bord de solutions positives de (0.1.1) définie ci-dessous.

Définition 1.2 *Soient $\mu_\delta \in \mathfrak{M}(\Sigma_\delta)$ pour tout $\delta \in (0, \delta^*)$ et $\mu \in \mathfrak{M}(\Sigma)$. On dit que $\mu_\delta \to \mu$ lorsque $\delta \to 0$ au sens faible des mesures si*

$$\lim_{\delta\to 0}\int_{\Sigma_\delta}\phi(\sigma(x))d\mu_\delta = \int_\Sigma \phi d\mu \qquad \forall\phi \in C_c(\Sigma). \tag{0.1.10}$$

Une fonction $u \in C(\Omega)$ *possède une trace au bord* $\mu \in \mathfrak{M}(\Sigma)$ *si*

$$\lim_{\delta \to 0} \int_{\Sigma_\delta} \phi(\sigma(x))u(x)dS = \int_\Sigma \phi d\mu \qquad \forall \phi \in C_c(\Sigma). \tag{0.1.11}$$

De façon analogue, si A *est un sous-ensemble relativement ouvert de* Σ*, on dit que* u *possède une trace au bord sur* A *au sens faible des mesures si* $\mu \in \mathfrak{M}(A)$ *et* (0.1.10) *reste valide pour tout* $\phi \in C_c(A)$*.*

La dichotomie suivante est obtenue par une combinaison des idées de [37], [42] et d'une construction géométrique de [3].

Théorème 1.3 *Supposons que* $g \in \mathcal{G}_0$ *satisfasse* (0.1.9) *ou que* g *soit une fonction continue et satisfasse*

$$\liminf_{r \to \infty} \frac{g(r)}{r^q} > 0 \tag{0.1.12}$$

où $1 < q \leq 2$*. Soit* $u \in C^2(\Omega)$ *une solution positive de* (0.1.1)*. Alors pour tout* $x_0 \in \partial\Omega$ *la dichotomie suivante a lieu*

(i) Ou bien il existe un voisinage ouvert U *de* x_0 *tel que*

$$\int_{\Omega \cap U} g(|\nabla u|)d(x)dx < \infty \tag{0.1.13}$$

et une mesure de Radon positive μ_U *sur* $\partial\Omega \cap U$ *telle que* $u|_{\Sigma_\delta \cap U}$ *converge vers* μ_U *au sens faible des mesures lorsque* $\delta \to 0$*.*

(ii) Ou bien pour tout voisinage ouvert U *de* x_0*,*

$$\int_{\Omega \cap U} g(|\nabla u|)d(x)dx = \infty, \tag{0.1.14}$$

et

$$\lim_{\delta \to 0} \int_{\Sigma_\delta \cap U} udS = \infty. \tag{0.1.15}$$

L'ensemble $\mathcal{S}(u)$ des points $x_0 \in \partial\Omega$ vérifiant la propriété (ii) est fermé et il existe une unique mesure de Radon positive μ sur $\mathcal{R}(u) := \partial\Omega \setminus \mathcal{S}(u)$ telle que $u|_{\Sigma_\delta}$ converge vers μ au sens faible des mesures sur $\mathcal{R}(u)$. Le couple $(\mathcal{S}(u), \mu)$ est appelé la trace au bord de u et noté $tr_{\partial\Omega}(u)$.

Réciproquement, étant donnée une mesure de Borel régulière (pas nécessairement localement bornée) ν sur $\partial\Omega$, on montre l'existence d'une solution de (0.1.2) dont la trace au bord est ν.

Théorème 1.4 *Supposons* $1 < q < q_c$*,* $\mathcal{S} \subsetneq \partial\Omega$ *fermé et* $\mu \in \mathfrak{M}_+(\mathcal{R})$ *où* $\mathcal{R} := \partial\Omega \setminus \mathcal{S}$*. Alors il existe une solution positive* u *de* (0.1.2) *telle que* $tr_{\partial\Omega}(u) = (\mathcal{S}, \mu)$*.*

L'ingrédient principal pour obtenir ce résultat est le résultat de stabilité du théorème 1.1 et l'estimation locale suivante :

Proposition 1.5 *Supposons* $1 < q < 2$, $U \subset \partial\Omega$ *relativement ouvert et* $\mu \in \mathfrak{M}_+^b(U)$. *Alors pour tout ensemble compact* $\Theta \subset \Omega$, *il existe* $C = C(N, q, \Omega, \Theta, \|\mu\|_{\mathfrak{M}(U)}) > 0$ *telle que toute solution positive* u *de* (0.1.2) *dans* Ω *de trace au bord* $(\mathcal{S}, \mu')$ *où* $\mathcal{S}$ *est fermé,* $U \subset \partial\Omega \setminus \mathcal{S} := \mathcal{R}$ *et* μ' *est une mesure de Radon positive sur* $\mathcal{R}$ *vérifiant* $\mu'|_U = \mu$, *on a*

$$u(x) \leq C \qquad \forall\, x \in \Theta. \tag{0.1.16}$$

Remaquons que l'hypothèse $\mathcal{S} \subsetneq \partial\Omega$ dans le théorème 1.4 est nécessaire car Alarcón, Garciá-Melián et Quaas [2] ont prouvé qu'il n'existe aucune *grande solution*, c'est-à-dire une solution qui explose partout sur $\partial\Omega$. Si $q_c \leq q < 2$, le théorème 1.4 reste vrai si $\mu = 0$ et $\mathcal{S} = \overline{G}$ où $G \subsetneq \partial\Omega$ est relativement ouvert et $\partial_{\partial\Omega} G$ satisfait la condition de sphère intérieure.

Pour caractériser des singularités isolées de solutions positives de (0.1.2), on considère le problème suivant dans l'hémisphère supérieur S_+^{N-1} dans $\mathbb{R}^N$

$$\begin{cases} -\Delta'\omega + \left(\left(\frac{2-q}{q-1}\right)^2 \omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} - \frac{2-q}{q-1}\left(\frac{q}{q-1} - N\right)\omega = 0 \quad \text{dans } S_+^{N-1} \\ \omega = 0 \quad \text{sur } \partial S_+^{N-1}, \end{cases} \tag{0.1.17}$$

où ∇' et Δ' désignent respectivement le gradient covariant et l'opérateur de Laplace-Beltrami sur S^{N-1}. A chaque solution ω de (0.1.17) on peut associer une solution séparable singulière u_s de (0.1.2) dans $\mathbb{R}_+^N := \{x = (x_1, x_2, ..., x_N) = (x', x_N) : x_N > 0\}$ s'annulant sur $\partial\mathbb{R}_+^N \setminus \{0\}$ qui est définie à l'aide des coordonnées sphériques $(r, \sigma) = (|x|, \frac{x}{|x|})$

$$u_s(x) = u_s(r, \sigma) = r^{-\frac{2-q}{q-1}}\omega(\sigma) \qquad \forall x \in \overline{\mathbb{R}_+^N} \setminus \{0\}. \tag{0.1.18}$$

Théorème 1.6 *Le problème* (0.1.17) *admet une solution positive unique, notée* ω_s, *si et seulement si* $1 < q < q_c$.

Grâce à la solution ω_s, on peut décrire le comportement asymptotique, lorsque $x \to 0$, de la fonction $u_{\infty,0} := \lim_{k\to 0} u_{k\delta_0}$ où $u_{k\delta_0}$ est la solution du problème (0.1.3) avec $h(r) = r^q$, $1 < q < q_c$ et $\mu = k\delta_0$. Plus clairement,

Théorème 1.7 *Supposons* $1 < q < q_c$ *et* $0 \in \partial\Omega$. *Alors* $u_{\infty,0}$ *est une solution positive de* (0.1.2) *dans* Ω, *continue dans* $\overline{\Omega} \setminus \{0\}$ *et s'annulant sur* $\partial\Omega \setminus \{0\}$. *De plus, on a*

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} = \sigma \in S_+^{N-1}}} |x|^{\frac{2-q}{q-1}} u_{\infty,0}(x) = \omega_s(\sigma), \tag{0.1.19}$$

localement uniformément sur S_+^{N-1}.

La relation (0.1.19) nous permet de montrer l'unicité de la solution positive de (0.1.2) de trace au bord $(\{0\}, 0)$, d'où la classification suivante :

Théorème 1.8 *Supposons* $1 < q < q_c$ *et que* $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ *soit une solution positive de* (0.1.2) *s'annulant sur* $\partial\Omega \setminus \{0\}$. *Alors la dichotomie suivante a lieu*

(i) Ou bien il existe $k \geq 0$ *telle que* $u = u_{k\delta_0}$ *résout* (0.1.3) *avec* $g(r) = r^q$, $\mu = k\delta_0$ *et*

$$u(x) = kP^{\Omega}(x, 0)(1 + o(1)) \qquad \textit{lorsque } x \to 0 \tag{0.1.20}$$

où P^{Ω} *est le noyau de Poisson dans* Ω.

(ii) Ou bien $u = u_{\infty,0}$ *et* (0.1.19) *a lieu.*

On donne par la suite une estimation inférieure pour des points singuliers sur la frontière.

Théorème 1.9 *Supposons* $1 < q < q_c$ *et que* u *soit une solution positive de* (0.1.2) *de trace au bord* $(\mathcal{S}(u), \mu)$. *Alors pour tout* $z \in \mathcal{S}(u)$, *on a*

$$u(x) \geq u_{\infty,z}(x) := \lim_{k\to\infty} u_{k\delta_z}(x) \qquad \forall x \in \Omega. \tag{0.1.21}$$

Le comportement asymptotique de $u_{\infty,z}$ *lorsque* $x \to z$ *est déterminé par* ω_s *à l'aide d'une translation et d'une rotation.*

0.1.2 Eliminabilité de singularités au bord

L'exposant q_c joue un rôle crucial puisque l'on a le résultat d'éliminabilité des singularités isolées au bord suivant :

Théorème 1.10 *Supposons* $q_c \leq q < 2$, *alors toute solution positive* $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ *de* (0.1.2) *s'annulant sur* $\partial\Omega \setminus \{0\}$ *est identiquement nulle.*

L'équation (0.1.2) peut être bien comprise dans le cas sur-critique à l'aide de la capacité $C_{\frac{2-q}{q},q'}$ en dimension $N - 1$, à condition que des solutions soient *modérées* ou *sigma-modérées*. Suivant Dynkin et Kuznetsov [14], [17], [25], on définit

Définition 1.11 *Une solution positive* u *de* (0.1.2) *est modérée s'il existe une mesure* $\mu \in \mathfrak{M}_+(\partial\Omega)$ *telle que* u *résout le problème* (0.1.3) *avec* $g(r) = r^q$. *Elle est appelée sigma-modérée s'il existe une suite croissante de mesures* $\mu_n \in \mathfrak{M}_+(\partial\Omega)$ *telle que la suite de solutions* $\{u_{\mu_n}\}$ *soit croissante et converge vers* u *localement uniformément dans* Ω *lorsque* $n \to \infty$.

Autrement dit, u est une solution modérée si et seulement si $u \in L^1(\Omega)$ et $|\nabla u| \in L^1_d(\Omega)$, ceci entraîne le résultat d'éliminabilité suivant :

Théorème 1.12 *Supposons* $q_c \leq q < 2$ *et que* $K \subset \partial\Omega$ *soit un compact et satisfasse* $C_{\frac{2-q}{q},q'}(K) = 0$. *Alors toute solution modérée positive* $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus K)$ *de* (0.1.2) *s'annulant sur* $\partial\Omega \setminus K$ *est identiquement nulle.*

Par conséquent, le résultat ci-dessus reste vrai si u est une solution sigma-modérée. Le théorème suivant nous donne une condition nécessaire pour résoudre le problème (0.1.3) avec $g(r) = r^q$, $q_c \leq q < 2$.

Théorème 1.13 *Supposons* $q_c \leq q < 2$ *et que* u *soit une solution modérée positive de* (0.1.2) *de trace au bord* $\mu \in \mathfrak{M}_+(\partial\Omega)$. *Alors* μ *est absolument continue par rapport à la capacité* $C_{\frac{2-q}{q},q'}$.

0.1.3 Eliminabilité de singularités à l'intérieur

On étudie aussi l'équation du type

$$-\Delta u + \tilde{g}(|\nabla u|) = \tilde{\mu} \qquad \text{dans } \Omega \tag{0.1.22}$$

et le problème de Dirichlet associé à (0.1.22)

$$\begin{cases} -\Delta u + \tilde{g}(|\nabla u|) &= \tilde{\mu} \quad \text{dans } \Omega \\ u &= 0 \quad \text{sur } \partial\Omega \end{cases} \tag{0.1.23}$$

où $\tilde{\mu}$ est une mesure de Radon bornée positive sur Ω et $\tilde{g}$ est une fonction localement Lipschitzienne, croissante, s'annulant en 0. Une fonction u est dite solution de (0.1.23) si $u \in L^1(\Omega)$, $\tilde{g}(|\nabla u|) \in L^1(\Omega)$ et

$$\int_\Omega (-u\Delta\zeta + \tilde{g}(|\nabla u|)\zeta)\, dx = \int_\Omega \zeta d\tilde{\mu} \tag{0.1.24}$$

pour tout $\zeta \in X(\Omega)$. La *condition d'intégrale sous-critique* est la suivante :

$$\int_1^\infty \tilde{g}(s) s^{-\frac{2N-1}{N-1}} ds < \infty. \tag{0.1.25}$$

Par un raisonnement analogue à celui de la preuve du théorème 1.1, on peut mettre en évidence l'existence d'une solution du problème (0.1.23) et d'un résultat de stabilité.

Théorème 1.14 *On suppose que* $\tilde{g}$ *satisfait* (0.1.25). *Alors pour toute mesure bornée* $\tilde{\mu} \in \mathfrak{M}_+(\Omega)$, *il existe une solution maximale* $\overline{u}_{\tilde{\mu}}$ *de* (0.1.23). *De plus, si* $\{\mu_n\}$ *est une suite de mesures bornées positives sur* Ω *convergeant vers une mesure bornée* $\tilde{\mu} \in \mathfrak{M}_+(\Omega)$ *au sens faible des mesures sur* Ω *et* $\{u_{\mu_n}\}$ *est une suite de solutions de* (0.1.23) *avec* $\tilde{\mu} = \mu_n$, *alors il existe une sous-suite* $\{\mu_{n_k}\}$ *telle que* $\{u_{\mu_{n_k}}\}$ *converge vers une solution* $u_{\tilde{\mu}}$ *de* (0.1.23) *dans* $L^1(\Omega)$ *et* $\{\tilde{g}(|\nabla u_{\mu_{n_k}}|)\}$ *converge vers* $\tilde{g}(|\nabla u_{\tilde{\mu}}|)$ *dans* $L^1(\Omega)$.

Dans le cas où $\tilde{g}$ est une fonction puissance,

$$-\Delta u + |\nabla u|^q = \tilde{\mu} \qquad \text{dans } \Omega \tag{0.1.26}$$

avec $1 < q < 2$, la valeur critique d'exposant est $q^* = \frac{N}{N-1}$. Dans le cas sous-critique $1 < q < q^*$, si $\mu \in \mathfrak{M}^b(\Omega)$, Barles et Porretta ont prouvé l'existence et l'unicité d'une solution de (0.1.23) (consulter [5] pour la résolution d'une classe d'équations beaucoup plus générale). Dans le cas sur-critique $q^* \leq q < 2$, à l'aide de la capacité $C_{1,q'}$ en dimension N, on peut établir un résultat d'éliminabilité d'ensembles singuliers intérieurs.

Théorème 1.15 *Supposons* $q^* \leq q < 2$ *et* $K \subset \Omega$ *compact. Si* $C_{1,q'}(K) = 0$ *alors toute solution positive* $u \in C^2(\overline{\Omega} \setminus K)$ *de*

$$-\Delta u + |\nabla u|^q = 0 \tag{0.1.27}$$

dans $\Omega \setminus K$ *vérifiant* $\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS$ *borné reste bornée et peut être prolongée en une solution de la même équation dans* Ω.

Finalement, on prouve la condition nécessaire suivante sous laquelle l'équation (0.1.26) est résolue.

Théorème 1.16 *Supposons* $q^* \leq q < 2$ *et* $\tilde{\mu} \in \mathfrak{M}_+(\Omega)$. *Soit* $u \in L^1(\Omega)$ *une solution de* (0.1.23) *avec* $\tilde{g}(r) = r^q$ *dans* Ω *vérifiant* $|\nabla u| \in L^q(\Omega)$. *Alors* $\tilde{\mu}(E) = 0$ *si* $C_{1,q'}(E) = 0$ *où* $E \subset \Omega$ *est un ensemble Borélien.*

0.2 Equations de la chaleur dégénérées non-linéaires

Dans les deux derniers chapitres, on étudie quelques propriétés locales et globales de solutions d'équations paraboliques du type

$$\partial_t u - \Delta_p u + f(u) = 0 \tag{0.2.1}$$

dans $Q_\infty := \mathbb{R}^N \times (0, \infty)$ $(N \geq 2)$ où $p > 1$, $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, $f : \mathbb{R} \to \mathbb{R}$ est une fonction continue, croissante, positive sur $(0, \infty)$ et satisfait $f(0) = 0$ et $\lim_{s\to\infty} f(s) = \infty$. En particulier, on étudie de manière très fine l'équation modèle suivante

$$\partial_t u - \Delta_p u + u^\alpha \ln^\beta(u + 1) = 0, \tag{0.2.2}$$

avec $\alpha > 0$ et $\beta > 0$, qui caractérise très bien l'*absorption faiblement sur-linéaire*.

La version elliptique de (0.2.2) (avec $p = 2$) a été traitée d'abord par Richard et Véron [45]. Ils ont en fait mis en évidence tous les comportements asymptotiques possibles d'une solution positive de

$$-\Delta u + u(\ln^+ u)^\gamma = 0 \tag{0.2.3}$$

dans $\Omega \setminus \{0\}$ $(0 \in \Omega)$ selon des positions relatives de γ et 2. Après, Fabbri et Licois [18] ont étudié le problème de trace au bord (au sens défini dans [37]) d'une solution de (0.2.3) dans une boule B_R et ont réussi à classifier complètement les solutions singulières en un point appartenant à $\partial\Omega$ en fonction des positions relatives de γ et 2.

Si $p = 2$ et $f(u) = |u|^{q-1} u$ avec $q > 1$, la structure d'ensemble des solutions de (0.2.1) est bien comprise et dépend des positions relatives de q et $q_1 := \frac{N+2}{N}$, la valeur d'exposant critique pour l'équation

$$\partial_t u - \Delta u + |u|^{q-1} u = 0. \tag{0.2.4}$$

On rappelle brièvement ci-dessous les résultats classiques concernant (0.2.4) dans la littérature.

- Dans le cas sous-critique $1 < q < q_1$, Brezis et Friedman [8] ont prouvé que pour tout $k \geq 0$ il existe une unique *solution fondamentale*, c'est-à-dire la solution positive $u := u_{k\delta_0} \in C(\overline{Q_\infty} \setminus \{(0,0)\}) \cap C^{2,1}(Q_\infty)$ du problème

$$\begin{cases} \partial_t u - \Delta u + |u|^{q-1} u &= 0 \quad \text{dans } Q_\infty \\ \qquad\qquad u(.,0) &= k\delta_0 \quad \text{dans } \mathcal{D}'(\mathbb{R}^N). \end{cases} \tag{0.2.5}$$

De plus, Brezis, Peletier et Terman [9] ont réussi à trouver une solution *très singulière* U_s de l'équation (0.2.4) dans Q_∞ vérifiant $u(x,0) = 0$ pour tout $x \neq 0$ sous la forme auto-similaire

$$U_s(x,t) = t^{-\frac{1}{p-1}}\Psi\left(\frac{|x|}{\sqrt{t}}\right) \tag{0.2.6}$$

où $\Psi(s)$ est la solution du problème

$$\begin{cases} \Psi'' + \left(\frac{N-1}{s} + \frac{s}{2}\right)\Psi' + \frac{1}{p-1}\Psi - \Psi^p = 0 \qquad \text{dans } (0,\infty) \\ \Psi > 0 \text{ sur } [0,\infty), \quad \Psi'(0) = 0 \\ \lim_{s\to\infty} s^{\frac{2}{p-1}}\Psi(s) = 0. \end{cases} \tag{0.2.7}$$

et le comportement asymptotique de Ψ lorsque $s \to \infty$ est donné par

$$\Psi(s) = C\, e^{-\frac{s^2}{4}} s^{\frac{2}{p-1}-N}[1 + O(s^{-2})] \tag{0.2.8}$$

où C est une constante positive. Le lien entre la solution très singulière et les solutions fondamentales est établi par Kamin et Peletier [21] : ils ont montré en fait que $U_s = \lim_{k\to\infty} u_{k\delta_0}$ localement uniformément dans $\overline{Q_\infty} \setminus \{(0,0)\}$. Après, l'équation (0.2.4) a été étudiée de manière plus générale par Marcus et Véron [39] à l'aide d'une notion de *trace initiale*. Ils ont prouvé que toute solution positive u de (0.2.4) possède une trace initiale définie par une mesure de Borel régulière (pas nécessairement localement bornée). Mieux encore, si $1 < q < q_1$, ils ont mis en évidence l'existence d'une bijection entre l'ensemble de telles mesures et l'ensemble des solutions positives de (0.2.4) dans $Q_T := \mathbb{R}^N \times (0,T)$.

- Dans le cas sur-critique $q \geq q_1$, Brezis et Friedman [8] ont prouvé que des singularités concentrées sur des sous-ensembles discrets sont éliminables. En utilisant des capacités de Bessel appropriées, Baras et Pierre [4] ont abouti à une caractérisation complète d'ensembles éliminables. De plus, ils ont donné des conditions nécessaires et suffisantes pour lesquelles une mesure de Radon (pas nécessairement positive) soit une trace initiale d'une solution de (0.2.4). Ensuite, Marcus et Véron [39] ont aussi fourni des conditions nécessaires et suffisantes d'existence d'une solution maximale de (0.2.4) avec une trace initiale donnée.

Si $p > 2$, Kamin et Vázquez [23] ont imposé quelques conditions sur f sous lesquelle il existe les solutions fondamentales $u_{k\delta_0}$ et la solution très singulière de l'équation (0.2.1). De plus, dans le cas où $f(u) = u^q$ avec $q > 1$, ils ont montré que la valeur critique est $q_2 = p - 1 + \frac{p}{N}$: la solution fondamentale $u := u_{k\delta_0}$ de

$$\begin{cases} \partial_t u - \Delta_p u + u^q &= 0 \quad \text{dans } Q_\infty \\ u(.,0) &= k\delta_0 \quad \text{dans } \mathcal{D}'(\mathbb{R}^N) \end{cases} \tag{0.2.9}$$

existe si $1 < q < q_2$ et la solution très singulière existe si $p - 1 < q < q_2$. Ce cadre est optimal car aucune solution très singulière n'existe si $1 < q \leq p - 1$ ou $q \geq q_2$.

Le cas où $1 < p < 2$ et $f(u) = u^q$ avec $q > 1$ a été traité par Chen, Qi et Wang dans [10], [11]. Comme dans le cas $p > 2$, si $q \geq q_2$, il est démontré qu'aucune solution singulière n'existe. Par contre, si $1 < q < q_2$, le phénomène se produit de manière différente, c'est-à-dire les solutions fondamentales et la solution très singulière existent à la fois.

L'étude de l'équation

$$\partial_t u - \Delta_p u + u^q = 0 \qquad \text{dans } Q_T^\Omega, \tag{0.2.10}$$

où Ω est un domaine de $\mathbb{R}^N$, a été effectuée en terme de trace initiale par Bidaut-Véron, Chasseigne et Véron [6]. Ils ont établi l'existence d'une trace initiale dans $\mathcal{B}_+^{reg}(\Omega)$ d'une *solution faible* pour des valeurs différentes de p et q (y compris le cas $0 < q \leq 1$) et ont classifié des traces initiales dans le cas où $p > 2$, $0 < q < p-1$ et $\Omega = \mathbb{R}^N$ ou Ω est bornée. Ils ont aussi étudié le problème de Cauchy associé avec donnée initiale dans $\mathcal{B}_+^{reg}(\Omega)$.

De manière analogue, notre but dans les deux derniers chapitres est de traiter les questions suivantes :

(a) l'existence des solutions fondamentales, c'est-à-dire les solutions dont la trace initiale est $k\delta_0$ avec $k > 0$ et le comportement de la fonction limite (si elle existe) des fonctions fondamentales lorsque $k \to \infty$;

(b) l'existence d'une trace initiale et la classification des traces initiales ;

(c) des résultats d'existence, d'unicité et de non-unicité pour le problème de Cauchy associé.

A la lumière des travaux de Kamin et Vázquez [23], on commence par étudier deux équations spécifiques qui découlent de (0.2.1). La première est l'équation différentielle ordinaire associée à (0.2.1)

$$\phi' + f(\phi) = 0. \tag{0.2.11}$$

Il est bien connu que si

$$\int_1^\infty \frac{ds}{f(s)} < \infty, \tag{0.2.12}$$

alors l'équation (0.2.11) admet une solution maximale ϕ_∞ définie sur $(0, \infty)$ qui explose en 0. En fait cette solution est la limite des fonctions ϕ_a lorsque $a \to \infty$ où ϕ_a est la solution de (0.2.11) dans $[0, \infty)$ avec la donnée initiale $\phi_a(0) = a$. De plus, elle est implicitement déterminée par la formule suivante

$$\int_{\phi_\infty(t)}^\infty \frac{ds}{f(s)} = t \qquad \forall t > 0. \tag{0.2.13}$$

Par contre, si (0.2.12) n'est pas vérifiée, une telle solution n'existe pas car $\lim_{a\to\infty} \phi_a = \infty$ dans $(0, \infty)$. Cette solution joue un rôle important puisqu'elle domine toute solution u de (0.2.1) vérifiant

$$\lim_{|x|\to\infty} u(x,t) = 0 \tag{0.2.14}$$

pour tout $t > 0$, localement uniformément sur $(0, \infty)$.

La deuxième équation issue (0.2.1) est l'équation stationnaire associée

$$-\Delta_p w + f(w) = 0. \tag{0.2.15}$$

Concernant cette équation, on considère la quantité suivante

$$\int_1^\infty \frac{ds}{F(s)^{\frac{1}{p}}}. \tag{0.2.16}$$

D'après des résultats de Keller-Osserman [24, Théorème III], [44] et de Vázquez [47], si

$$\int_1^\infty \frac{ds}{F(s)^{\frac{1}{p}}} < \infty, \tag{0.2.17}$$

alors l'équation (0.2.15) admet une solution maximale $W_{\mathbb{R}^N_*}$ dans $\mathbb{R}^N \setminus \{0\}$. Cette solution est la limite, lorsque $R \to \infty$ et $\epsilon \to 0$ successivement, des solutions $W := W_{\epsilon,R}$ de (0.2.15) dans $\Gamma_{\epsilon,R} := B_R \setminus \overline{B}_\epsilon$ vérifiant $\lim_{|x|\downarrow\epsilon} W_{\epsilon,R}(x) = \infty$ et $\lim_{|x|\uparrow R} W_{\epsilon,R}(x) = \infty$. Au contraire, si (0.2.17) n'a pas lieu, de telles solutions $W_{\epsilon,R}$ et $W_{\mathbb{R}^N_*}$ n'existent pas, ceci permet de prouver l'existence de solutions globales de (0.2.15) dans $\mathbb{R}^N$.

Une condition additionnelle portant sur f est l'*additivité sur-critique*

$$f(s+s') \geq f(s) + f(s') \quad \forall s, s' \geq 0, \tag{0.2.18}$$

ce qui, combinée avec la monotonicité de f, implique

$$\liminf_{s\to\infty} \frac{f(s)}{s} > 0. \tag{0.2.19}$$

Il convient de souligner que si $p \geq 2$ les conditions (0.2.17) et (0.2.19) impliquent (0.2.12). Par contre, si $1 < p < 2$, cette implication n'est plus valable. Sous les conditions (0.2.12) et (0.2.17), on peut établir des estimations universelles pour les solutions singulières de (0.2.1), ce qui éclaire notre étude de la structure d'ensemble de telles solutions.

0.2.1 Singularités isolées

Dans cette section on se concentre sur les singularités isolées. Kamin et Vázquez [23, Lemmes 2.3 et 2.4] ont prouvé que si $p > 2$ et si f satisfait la *condition de singularité faible*

$$\int_1^\infty s^{-p-\frac{p}{N}} f(s) ds < \infty, \tag{0.2.20}$$

alors pour tout $k > 0$, il existe une unique solution positive $u := u_{k\delta_0}$ de

$$\begin{cases} \partial_t u - \Delta_p u + f(u) &= 0 \quad \text{dans } Q_\infty \\ u(.,0) &= k\delta_0 \quad \text{dans } \mathbb{R}^N. \end{cases} \tag{0.2.21}$$

En outre, l'application $k \mapsto u_{k\delta_0}$ est croissante. Leur méthode repose essentiellement sur le fait que la solution fondamentale $v := v_{k\delta_0}$ (ou solution de Barenblatt-Prattle) [22], [23] de

$$\begin{cases} \partial_t v - \Delta_p v &= 0 \quad \text{dans } Q_\infty \\ v(.,0) &= k\delta_0 \quad \text{dans } \mathbb{R}^N, \end{cases} \tag{0.2.22}$$

est à support compact dans certaine boule $B_{\delta_k(t)}$ où $\delta_k(t)$ dépend de N et p et peut être explicitement déterminé. Puisque $v_{k\delta_0}$ est une sur-solution de (0.2.21), la condition (0.2.20) implique $f(v_{k\delta_0}) \in L^1(B_R \times (0,T))$ pout tout $R, T > 0$. Dès que $\frac{2N}{N+1} < p \leq 2$, $v_{k\delta_0}(x,t) > 0$ pour tout $(x,t) \in Q_\infty$. Il est montré dans [41] que si $p = 2$, la condition (0.2.20) entraîne $f(v_{k\delta_0}) \in L^1(Q_T)$ pout tout $T > 0$. On prouve que ce résultat reste vrai si $\frac{2N}{N+1} < p < 2$ et plus précisément :

Théorème 2.1 *Supposons $p > \frac{2N}{N+1}$ et f satisfaisant à* (0.2.20). *Alors il existe une unique solution positive $u := u_{k\delta_0}$ de* (0.2.21).

Puisque $k \mapsto u_{k\delta_0}$ est croissante, il est naturel d'étudier la limite $\lim_{k\to\infty} u_{k\delta_0}$. Pour cela, on désigne par $\mathcal{U}_0$ l'ensemble des solutions positives u de (0.2.1) dans Q_∞ qui sont continues dans $\overline{Q_\infty} \setminus \{(0,0)\}$, s'annulent sur $\{(x,0) : x \neq 0\}$ et satisfont

$$\lim_{t\to 0} \int_{B_\epsilon} u(x,t)dx = \infty \qquad (0.2.23)$$

pour tout $\epsilon > 0$.

Théorème 2.2 *Supposons $p > \frac{2N}{N+1}$ et f satisfaisant les conditions* (0.2.12), (0.2.17) *et* (0.2.20). *Alors $\underline{U} := \lim_{k\to\infty} u_{k\delta_0}$ est un élément minimal de $\mathcal{U}_0$.*

Lorsque f ne satisfait pas (0.2.12) ou (0.2.17), le problème devient beaucoup plus compliqué. Les cas $f(u) = u^q$ et $f(u) = u^\alpha \ln^\beta(u+1)$ sont bien compris. En particulier :

(A) Si $f(u) = u^q$ avec $q > 0$ alors (0.2.12) est vérifiée si et seulement si $q > 1$ tandis que (0.2.17) est vérifiée si et seulement si $q > p - 1$. De plus, (0.2.20) est satisfaite si et seulement si $q < p - 1 + \frac{p}{N}$.

(B) Si $f(u) = u^\alpha \ln^\beta(u+1)$ $(\alpha, \beta > 0)$, alors (0.2.12) est vérifiée si et seulement si $\alpha > 1$ et $\beta > 0$, ou $\alpha = 1$ et $\beta > 1$ tandis que (0.2.17) est vérifiée si et seulement si $\alpha > p - 1$ et $\beta > 0$, ou $\alpha = p-1$ et $\beta > p$. De plus (0.2.20) est satisfaite si et seulement si $\alpha < p - 1 + \frac{p}{N}$ et $\beta > 0$.

Dès que $p \geq 2$, les phénomènes suivants se produisent en fonction des valeurs de α et β.

Théorème 2.3 *Supposons $p = 2$ et que $f(u) = u \ln^\beta(u+1)$ avec $\beta > 0$. Soit $u_{k\delta_0}$ la solution fondamentale de* (0.2.21). *Alors*

(i) Si $0 < \beta \leq 1$, $\lim_{k\to\infty} u_{k\delta_0} = \infty$ dans Q_∞.

(ii) Si $1 < \beta \leq 2$, $\lim_{k\to\infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ pour tout $(x,t) \in Q_\infty$ où ϕ_∞ est la solution maximale de (0.2.11).

Théorème 2.4 *On suppose que $p > 2$ et que $f(u) = u^\alpha \ln^\beta(u+1)$ où $\alpha \in (1, p-1)$ et $\beta > 0$. Soit $u_{k\delta_0}$ la solution de* (0.2.21). *Alors $\lim\limits_{k\to\infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ pour tout $(x,t) \in Q_\infty$.*

Théorème 2.5 *On suppose que $p > 2$ et que $f(u) = u \ln^\beta(u+1)$ avec $\beta > 0$. Soit $u_{k\delta_0}$ la solution de* (0.2.21). *Alors*

(i) Si $\beta > 1$ alors $\lim\limits_{k\to\infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ pour tout $(x,t) \in Q_\infty$,

(ii) Si $0 < \beta \leq 1$ *alors* $\lim_{k\to\infty} u_{k\delta_0}(x,t) = \infty$ *pour tout* $(x,t) \in Q_\infty$.

0.2.2 Trace initiale

0.2.2.1 Cas $p = 2$

Grâce aux résultats ci-dessus, on développe une *nouvelle construction de trace initiale* de solutions positives, localement bornées de (0.2.1) dans Q_∞. Il convient de noter que dans le cas $f(u) = |u|^{q-1} u$, la trace initiale a été construite [39] à l'aide de combinaisons de l'inégalité de Hölder et d'un choix délicat de fonctions tests. De manière très différente, notre nouvelle méthode repose sur le principle de maximum combiné ou bien avec la condition de Keller-Osserman (0.2.17), ou bien avec des propriétés de $\lim_{k\to\infty} u_{k\delta_0}$ si (0.2.17) n'est pas vérifiée. On montre d'abord le :

Théorème 2.6 *Soit* $u \in C^{2,1}(Q_\infty)$ *une solution positive de* (0.2.1) *dans* Q_∞*. L'ensemble* $\mathcal{R}(u)$ *des points* $z \in \mathbb{R}^N$ *tels qu'il existe une boule ouverte* $B_r(z)$ *telle que* $f(u) \in L^1(Q_T^{B_r(z)})$ *est un sous-ensemble ouvert. De plus, il existe une mesure de Radon positive* $\mu := \mu(u)$ *sur* $\mathcal{R}(u)$ *vérifiant*

$$\lim_{t\to 0} \int_{\mathcal{R}(u)} u(x,t)\zeta(x)dx = \int_{\mathcal{R}(u)} \zeta(x)d\mu(x) \qquad \forall \zeta \in C_c(\mathcal{R}(u)). \tag{0.2.24}$$

Cette proposition nous permet de définir la trace initiale d'une solution positive de (0.2.1).

Définition 2.7 *Le couple* $(\mathcal{S}(u), \mu)$ *où* $\mathcal{S}(u) = \mathbb{R}^N \setminus \mathcal{R}(u)$ *est appelé la trace initiale de u dans* Ω *et noté* $tr_{\mathbb{R}^N}(u)$*. L'ensemble* $\mathcal{R}(u)$ *est appelé l'ensemble régulier de la trace initiale de u et la mesure* μ *est appelée la partie régulière de la trace initiale. L'ensemble* $\mathcal{S}(u)$ *est fermé et est appelé la partie singulière de la trace initiale de u.*

Il est intéressant de noter que si f satisfait (0.2.17) et $z \in \mathcal{S}(u)$, alors pour tout voisinage ouvert U de z, on a

$$\lim_{t\to 0} \int_U u(x,t)dx = \infty. \tag{0.2.25}$$

0.2.2.2 Cas $p \geq 2$

Le cas de la puissance $f(u) = u^q$ avec $q > 1$ a été traité par Bidaut-Véron, Chasseigne et Véron dans [6]. Néanmoins, leur méthode repose essentiellement sur le fait que le terme non-linéaire est une fonction puissance, ceci permet d'utiliser l'inégalité de Hölder pour prouver la domination du terme d'absorption sur d'autres termes. Dans notre cas, en combinant l'idée de [6], des techniques utilisées dans le chapitre 2 pour établir l'existence d'une trace initiale, un résultat de stabilité de [46, Théorème 2], [31, Théorème 1.1] et l'inégalité de Harnack de [12], on démontre le résultat suivant :

Théorème 2.8 *Supposons* $p \geq 2$ *et que* f *satisfasse* (0.2.20)*. Soit* $u \in C(Q_T)$ *une solution faible positive de* (0.2.1) *dans* Q_T*. Alors pour tout* $y \in \mathbb{R}^N$ *la dichotomie suivante a lieu* *(i) Ou bien*

$$u(x,t) \geq \lim_{k\to\infty} u_{k\delta_0}(x-y,t) \qquad \forall (x,t) \in Q_T, \tag{0.2.26}$$

(ii) Ou bien il existe un voisinage ouvert U de y et une mesure de Radon $\mu_U \in \mathfrak{M}_+(U)$ tels que

$$\lim_{t\to 0} \int_U u(x,t)\zeta(x)dx = \int_U \zeta d\mu_U \quad \forall \zeta \in C_c(U). \tag{0.2.27}$$

En fait, si (0.2.20) est vérifiée, (0.2.26) est équivalente au fait que pour tout voisinage ouvert U de y, on a

$$\limsup_{t\to 0} \int_U u(x,t)dx = \infty. \tag{0.2.28}$$

Cependant, si (0.2.20) n'est pas vérifiée, on n'a que (0.2.26) $\Longrightarrow$ (0.2.28).

Il convient de noter que ce résultat est nouveau même dans le cas $p = 2$. L'ensemble des points y tels que (0.2.27) (resp. (0.2.28)) ait lieu est ouvert (resp. fermé) et est noté $\mathcal{R}(u)$ (resp. ($\mathcal{S}(u)$)). En utilisant une partition de l'unité, on montre qu'il existe une unique mesure de Radon $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ telle que

$$\lim_{t\to 0} \int_{\mathcal{R}(u)} u(x,t)\zeta(x)dx = \int_{\mathcal{R}(u)} \zeta d\mu \quad \forall \zeta \in C_c(\mathcal{R}(u)). \tag{0.2.29}$$

Grâce au résultat ci-dessus, on peut alors définir la trace initiale d'une solution positive u de (0.2.1) dans Q_T.

Définition 2.9 *Le couple $(\mathcal{S}(u), \mu)$ est appelé la trace initiale de la solution u et noté $tr_{\mathbb{R}^N}(u)$. L'ensemble $\mathcal{S}(u)$ est l'ensemble des points singuliers de $tr_{\mathbb{R}^N}(u)$, tandis que μ est la partie régulière de $tr_{\mathbb{R}^N}(u)$.*

Comme dans [39], la trace initiale peut aussi être représentée par une mesure de Borel régulière, pas nécessairement localement bornée, c'est-à-dire qu'il existe une bijection entre $\mathcal{B}_+^{reg}(\mathbb{R}^N)$ et l'ensemble de couples :

$$CM_+(\mathbb{R}^N) = \left\{(\mathcal{S}, \mu) : \mathcal{S} \subset \mathbb{R}^N \text{ fermé}, \mu \in \mathfrak{M}_+(\mathcal{R}) \text{ avec } \mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}\right\}. \tag{0.2.30}$$

La mesure de Borel $\nu \in \mathcal{B}_+^{reg}(\mathbb{R}^N)$ correspondant à un couple $(\mathcal{S}, \mu) \in CM_+(\mathbb{R}^N)$ est déterminée par

$$\nu(A) = \begin{cases} \infty & \text{si } A \cap \mathcal{S} \neq \emptyset \\ \mu(A) & \text{si } A \subseteq \mathcal{R}, \end{cases} \qquad \forall A \subset \mathbb{R}^N, A \text{ Borélien}. \tag{0.2.31}$$

Dans tout ce qui suit, si u est une solution de (0.2.1), on utilisera la notation $tr_{\mathbb{R}^N}(u)$ (resp. $Tr_{\mathbb{R}^N}(u)$) pour la trace considérée comme un élément de $CM_+(\mathbb{R}^N)$ (resp. $\mathcal{B}_+^{reg}(\mathbb{R}^N)$).

D'après ce qui pécède, on obtient :

Théorème 2.10 *Supposons $p \geq 2$ et que f satisfasse* (0.2.20). *Soit u une solution positive de* (0.2.1). *Alors elle possède une trace initiale $\nu \in \mathcal{B}_+^{reg}(\mathbb{R}^N)$.*

Remarquons que dans le cas $p = 2$, ce résultat reste valable si la condition (0.2.20) est remplacée par la condition de Keller-Osserman (0.2.17). Le comportement de la limite des fonctions $u_{k\delta_0}$ lorsque $k \to \infty$ (où $u_{k\delta_0}$ est la solution de (0.2.21)) nous permet de décrire plus précisément la trace initiale.

Théorème 2.11 *On suppose que $p \geq 2$ et que f satisfait* (0.2.20) *et* (0.2.12)*. De plus, on suppose que $\lim_{k\to\infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ pour tout $(x,t) \in Q_\infty$ où ϕ_∞ est la solution maximale de* (0.2.15)*. Si u est une solution positive de* (0.2.1) *dans Q_∞, elle possède alors une trace initiale qui est ou bien une mesure de Borel ν_∞ vérifiant $\nu_\infty(\mathcal{O}) = \infty$ pour tout ouvert non-vide $\mathcal{O} \subset \mathbb{R}^N$, ou bien une mesure de Radon positive μ sur $\mathbb{R}^N$. Ce résultat est valabe en particulier si $f(u) = u^\alpha \ln^\beta(u+1)$ avec $p = 2$, $\alpha = 1$, $1 < \beta \leq 2$ ou $p > 2$, $1 < \alpha < p-1$, $\beta > 0$ ou $p > 2$, $\alpha = 1$, $\beta > 1$.*

La conséquence suivante découle du théorème 2.11.

Proposition 2.12 *Supposons que les hypothèses énoncées dans le théorème 2.11 soient satisfaites. De plus, on suppose que f satisfait* (0.2.18) *mais ne satisfait pas* (0.2.17)*. Alors, pour tout $a > 0$, il existe une solution positive $u \in C(Q_\infty)$ de* (0.2.1) *vérifiant*

$$\max\{\phi_\infty(t), w_a(|x|)\} \leq u(x,t) \leq \phi_\infty(t) + w_a(|x|) \qquad \forall (x,t) \in Q_\infty \tag{0.2.32}$$

où w_a est la solution de (0.2.33)*. Par conséquent, il y a un nombre infini de solutions de* (0.2.1) *avec la même trace au bord ν_∞. De plus, ϕ_∞ est la solution minimale parmi de telles solutions.*

On envisage ensuite le cas où $\lim_{k\to\infty} u_{k\delta_0} = \infty$ dans Q_∞.

Théorème 2.13 *Supposons que f satisfasse* (0.2.20) *mais ne satisfasse pas* (0.2.12)*. De plus, on suppose que $\lim_{k\to\infty} u_{k\delta_0} = \infty$ dans Q_∞. Si u est une solution positive de* (0.2.1) *dans Q_∞, alors u possède une trace initiale $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$. Ce résultat reste valable en particulier si $f(u) = u^\alpha \ln^\beta(u+1)$ avec $p \geq 2$, $\alpha = 1$ et $0 < \beta \leq 1$.*

Pour établir ce résultat, les outils fondamentaux sont : une méthode développée de [42], le résultat de stabilité et les théorèmes 2.3 et 2.5.

0.2.3 Problème de Cauchy avec donnée initiale non bornée

On étudie l'ensemble des solutions positives, localement bornées de (0.2.1) dans Q_∞, qui varie selon des hypothèses sur f. Pour cela, les solutions radiales de l'équation (0.2.15) représentent une aide efficace. Le résultat suivant repose sur le théorème de point fixe de Picard-Lipschitz et un résultat de Guedda et Véron [20, Théorème 5.2].

Proposition 2.14 *On suppose que $p > 1$ et que f est localement Lipschitzienne et ne satisfait pas la condition* (0.2.17)*. Pour tout $a > 0$, il existe une unique solution positive $w := w_a$ du problème*

$$\begin{cases} -(r^{N-1}|w'|^{p-2}w')' + r^{N-1}f(w) &= 0 \qquad \text{dans } \mathbb{R}_+ \\ w'(0) = 0, \qquad w(0) &= a. \end{cases} \tag{0.2.33}$$

Elle est déterminée par la formule suivante

$$w_a(r) = a + \int_0^r H_p\left(s^{1-N}\int_0^s \tau^{N-1} f(w_a(\tau))d\tau\right) ds \qquad (0.2.34)$$

où H_p est la fonction réciproque de la fonction $t \mapsto |t|^{p-2}t$.

Ce résultat étend un résultat de Vázquez et Véron [48] pour le cas $p = 2$ au cas général $p > 1$. Une conséquence remarquable de la proposition 2.14 est le résultat de non-unicité suivant :

Théorème 2.15 *On suppose que $p > \frac{2N}{N+1}$ et que f est localement Lipschitzienne, satisfait* (0.2.12) *mais ne satisfait pas* (0.2.17). *Pour toute fonction $u_0 \in C(Q_\infty)$ vérifiant*

$$w_a(|x|) \le u_0(x) \le w_b(|x|) \quad \forall x \in \mathbb{R}^N \qquad (0.2.35)$$

pour certain $0 < a < b$, alors il y a au moins deux solutions $\underline{u}, \overline{u} \in C(\overline{Q_\infty})$ de (0.2.1) *avec la même donnée initiale u_0. De plus, elles satisfont respectivement*

$$0 \le \underline{u}(x,t) \le \min\{w_b(|x|), \phi_\infty(t)\} \quad \forall (x,t) \in Q_\infty,$$

donc $\lim_{t\to\infty} \underline{u}(x,t) = 0$, uniformément par rappor à $x \in \mathbb{R}^N$, et

$$w_a(|x|) \le \overline{u}(x,t) \le w_b(|x|) \quad \forall (x,t) \in Q_\infty$$

donc $\lim_{|x|\to\infty} \overline{u}(x,t) = \infty$, uniformément par rapport à $t \ge 0$.

Dans le cas $p = 2$, si deux solutions de (0.2.1) ont la même donnée initiale et le même comportement asymptotique à l'infini, alors elles coïncident.

Théorème 2.16 *On suppose que $p = 2$ et que f satisfait* (0.2.18) *mais ne satisfait pas* (0.2.17). *Soient $u, \tilde{u} \in C(\overline{Q_\infty}) \cap C^{2,1}(Q_\infty)$ deux solutions positives de* (0.2.1) *avec donnée initiale $u_0 \in C(\mathbb{R}^N)$. Si pour tout $\epsilon > 0$,*

$$u(x,t) - \tilde{u}(x,t) = o(w_\epsilon(|x|)) \quad \textit{lorsque } x \to \infty \qquad (0.2.36)$$

localement uniformément par rapport à $t \ge 0$, alors $u = \tilde{u}$.

Au contraire, si f satisfait (0.2.17), étant donnée une fonction $u_0 \in C(\mathbb{R}^N)$, le théorème suivant affirme l'existence et l'unicité d'une solution continue de (0.2.1) avec la donnée initiale u_0. L'affirmation reste vraie si $C(\mathbb{R}^N)$ est remplacé par $\mathfrak{M}_+(\mathbb{R}^N)$.

Théorème 2.17 *Supposons $p = 2$ et que f satisfasse* (0.2.17) *et* (0.2.18). *Alors*

(i) Etant donnée une fonction positive $u_0 \in C(\mathbb{R}^N)$ il existe une unique solution positive $u \in C(\overline{Q_\infty})$ de (0.2.1) *dans Q_∞ avec la donnée initiale u_0.*

(ii) Etant donnée une mesure $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$, il existe au plus une solution positive $u \in C(Q_\infty)$ de (0.2.1) *dans Q_∞ telle que $f(u) \in L^1_{loc}(\overline{Q_\infty})$ et*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N} \zeta(x)d\mu(x) \quad \forall \zeta \in C_c(\mathbb{R}^N). \qquad (0.2.37)$$

De manière générale, en employant des raisonnements semblables à ceux de [39, Théorème 3.4], on établit l'existence d'une solution maximale et d'une solution minimale de (0.2.1) avec une trace initiale donnée appartenant à $CM_+(\mathbb{R}^N)$.

Théorème 2.18 *Supposons* $p = 2$ *et que* f *satisfasse* (0.2.17), (0.2.18) *et* (0.2.20). *Alors pour tout couple* $(\mathcal{S}, \mu) \in CM_+(\mathbb{R}^N)$, *il existe une solution maximale* $\overline{u}_{\mathcal{S},\mu}$ *et une solution minimale* $\underline{u}_{\mathcal{S},\mu}$ *de* (0.2.1) *dans* Q_∞ *avec la même trace initale* $(\mathcal{S}, \mu)$, *au sens suivant :*

$$\underline{u}_{\mathcal{S},\mu} \leq v \leq \overline{u}_{\mathcal{S},\mu} \tag{0.2.38}$$

pour toute solution positive $v \in C^{2,1}(Q_\infty)$ *de* (0.2.1) *dans* Q_∞ *telle que* $tr_{\mathbb{R}^N}(v) = (\mathcal{S}, \mu)$.

Remarquons que si $p = 2$ et $f(u) = |u|^{q-1}u$ avec $1 < q < q_1$, le comportement asymptotique précis de $\underline{U}$ lorsque $t \to 0$ permet de prouver l'unicité de la solution avec trace initiale donnée dans $CM_+(\mathbb{R}^N)$. En général, même si $p = 2$ et $f(u) = u \ln^\beta(u+1)$ avec $\beta > 2$, l'unicité reste une question ouverte. Cependant, si $\mathcal{S} = \emptyset$, l'unicité est mis en évidence comme énoncée plus haut dans le Théorème 2.17 (ii).

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Chapitre 1

Boundary trace and removable singularities of solutions to elliptic Hamilton-Jacobi equations

Abstract

We study the boundary value problem with measures for (E1) $-\Delta u + g(|\nabla u|) = 0$ in a bounded domain Ω in $\mathbb{R}^N$, satisfying (E2) $u = \mu$ on $\partial\Omega$ and prove that if $g \in L^1(1, \infty; t^{-(2N+1)/N} dt)$ is nondecreasing (E1)-(E2) can be solved with any positive bounded measure. When $g(r) \geq r^q$ with $q > 1$ we prove that any positive solution of (E1) admits a boundary trace which is an outer regular Borel measure, not necessarily bounded. When $g(r) = r^q$ with $1 < q < q_c = \frac{N+1}{N}$ we prove the existence of a positive solution with a general outer regular Borel measure $\nu \not\equiv \infty$ as boundary trace and characterize the boundary isolated singularities of positive solutions. When $g(r) = r^q$ with $q_c \leq q < 2$ we prove that a necessary condition for solvability is that μ must be absolutely continuous with respect to the Bessel capacity $C_{\frac{2-q}{q}, q'}$. We also characterize boundary removable sets for moderate and sigma-moderate solutions.

1.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary and $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing continuous function vanishing at 0. In this article we investigate several boundary data questions associated to nonnegative solutions of the following equation

$$-\Delta u + g(|\nabla u|) = 0 \qquad \text{in } \Omega, \tag{1.1.1}$$

and we emphasize on the particular case of

$$-\Delta u + |\nabla u|^q = 0 \qquad \text{in } \Omega. \tag{1.1.2}$$

where q is a real number mainly in the range $1 < q < 2$. We investigate first the generalized boundary value problem with measure associated to (1.1.1)

$$\begin{cases} -\Delta u + g(|\nabla u|) = 0 & \text{in } \Omega \\ u = \mu & \text{on } \partial\Omega \end{cases} \tag{1.1.3}$$

where μ is a measure on $\partial\Omega$. By a solution we mean an integrable function u such that $g(|\nabla u|) \in L^1_d(\Omega)$ where $d = d(x) := \text{dist}\,(x, \partial\Omega)$ satisfying

$$\int_\Omega (-u\Delta\zeta + g(|\nabla u|)\zeta)\,dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu \tag{1.1.4}$$

for all $\zeta \in X(\Omega) := \{\phi \in C^1_0(\overline{\Omega}) : \Delta\phi \in L^\infty(\Omega)\}$, where $\mathbf{n}$ denotes the normal outward unit vector to $\partial\Omega$. The *integral subcriticality condition* for g is the following

$$\int_1^\infty g(s)s^{-\frac{2N+1}{N}} ds < \infty. \tag{1.1.5}$$

When $g(r) \leq r^q$, this condition is satisfied if $0 < q < q_c := \frac{N+1}{N}$. Our main existence result is the following :

Theorem 1.1.1 *Assume g satisfies* (1.1.5). *Then for any positive bounded Borel measure μ on $\partial\Omega$ there exists a maximal positive solution $\overline{u}_\mu$ to problem* (1.1.3). *Furthermore the problem is closed for weak convergence of boundary data.*

Note that we do not know if problem (1.1.4) has a unique solution, *except if $g(r) = r^q$ with $0 < q < q_c$ and $\mu = c\delta_0$ in which case we prove that uniqueness holds.* A natural way for studying (1.1.1) is to introduce the notion of *boundary trace*. When $g(r) \geq r^q$ with $q > 1$ we prove in particular that the following result holds in which statement we denote $\Sigma_\delta = \{x \in \Omega : d(x) = \delta\}$ for $\delta > 0$:

Theorem 1.1.2 *Let u be any positive solution of* (1.1.1). *Then for any $x_0 \in \partial\Omega$ the following dichotomy occurs :*

(i) Either there exists an open neighborhood U of x_0 such that

$$\int_{\Omega\cap U} g(|\nabla u|)d(x)dx < \infty \tag{1.1.6}$$

and there exists a positive Radon measure μ_U on $\partial\Omega \cap U$ such that $u|_{\Sigma_\delta}$ converges to μ_U in the weak sense of measures when $\delta \to 0$.

(ii) Or for any open neighborhood U of x_0 there holds

$$\int_{\Omega\cap U} g(|\nabla u|)d(x)dx = \infty, \tag{1.1.7}$$

and

$$\lim_{\delta\to 0}\int_{\Sigma_\delta\cap U} udS = \infty. \tag{1.1.8}$$

The set $\mathcal{S}(u)$ of boundary points x_0 with the property (ii) is closed and there exists a unique positive Radon measure μ on $\mathcal{R}(u) := \partial\Omega \setminus \mathcal{S}(u)$ such that $u|_{\Sigma_\delta}$ converges to μ in the weak sense of measures on $\mathcal{R}(u)$. The couple $(\mathcal{S}(u), \mu)$ is the boundary trace of u, denoted by $tr_{\partial\Omega}(u)$. The trace framework has also the advantage of pointing out some of the main questions which remain to be solved as it was done for the semilinear equation

$$-\Delta u + h(u) = 0 \qquad \text{in } \Omega. \tag{1.1.9}$$

and the associated Dirichlet problem with measure

$$\left\{\begin{array}{rl} -\Delta u + h(u) = 0 & \text{in } \Omega \\ u = \mu & \text{on } \partial\Omega, \end{array}\right. \tag{1.1.10}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function vanishing at 0. Much is known since the first paper of Gmira and Véron [16] and many developments are due to Marcus and Véron [29]–[32] in particular when (1.1.9) is replaced by

$$-\Delta u + |u|^{q-1}u = 0 \qquad \text{in } \Omega \tag{1.1.11}$$

with $q > 1$. We recall below some of the main aspects of the results dealing with (1.1.9)–(1.1.11), this will play the role of the breadcrumbs trail for our study.

- Problem (1.1.10) can be solved (in a unique way) for any bounded measure μ if h satisfies

$$\int_1^\infty (h(s) + |h(-s)|)s^{-\frac{2N}{N-1}}ds < \infty. \tag{1.1.12}$$

If $h(u) = |u|^{q-1}u$ the condition (1.1.12) is verified if and only if $1 < q < q_s$, *the subcritical range*; $q_s = \frac{N+1}{N-1}$ is a critical exponent for (1.1.11).

- When $1 < q < q_s$, boundary isolated singularities of nonnegative solutions of (1.1.11) can be completely characterized i.e. if $u \in C(\overline{\Omega} \setminus \{0\})$ is a nonnegative solution of (1.1.11) vanishing on $\partial\Omega \setminus \{0\}$, then either it solves the associated Dirichlet problem with $\mu = c\delta_0$ for some $c \geq 0$ (*weak singularity*), or [32]

$$u(x) \approx Cd(x)|x|^{-\frac{q+1}{q-1}} \qquad \text{as } x \to 0. \quad (\textit{strong singularity}) \tag{1.1.13}$$

- Always in the subcritical range it is proved that for any couple $(\mathcal{S},\mu)$ where $\mathcal{S}\subset\partial\Omega$ is closed and μ is a positive Radon measure on $\mathcal{R}=\partial\Omega\setminus\mathcal{S}$ there exists a unique positive solution u of (1.1.11) with boundary trace $(\mathcal{S},\mu)$ (in the sense defined in Theorem 1.1.2).

- When $q\geq q_s$, i.e. the *supercritical range*, any solution $u\in C(\overline{\Omega}\setminus\{0\})$ of (1.1.11) vanishing on $\partial\Omega\setminus\{0\}$ is identically 0, i.e. *isolated boundary singularities are removable.* This result due to Gmira-Véron has been extended, either by probabilistic tools by Le Gall [20], [21], Dynkin and Kuznetsov [12], [13], with the restriction $q_s\leq q\leq 2$, or by purely analytic methods by Marcus and Véron [29], [30] in the whole range $q_s\leq q$. The key tool for describing the problem is the Bessel capacity $C_{\frac{2}{q},q'}$ in dimension $N-1$. We list some of the most striking results. The associated Dirichlet problem can be solved with $\mu\in\mathfrak{M}(\partial\Omega)$ if and only if μ is absolutely continuous with respect to the $C_{\frac{2}{q},q'}$-capacity. If $K\subset\partial\Omega$ is compact and $u\in C(\overline{\Omega}\setminus K)$ is a solution of (1.1.11) vanishing on $\partial\Omega\setminus K$, then u is identically zero if and only if $C_{\frac{2}{q},q'}(K)=0$. The complete characterization of positive solutions of (1.1.11) has been obtained by Mselati [28] when $q=2$, Dynkin [11] when $q_s\leq q\leq 2$, and finally by Marcus [27] when $q_s\leq q$; they proved in particular that any positive solution u is *sigma-moderate*, i.e. that there exists an increasing sequence of measures $\mu_n\in\mathfrak{M}_+(\partial\Omega)$ such that the sequence of the solutions $u=u_{\mu_n}$ of the associated Dirichlet problem with $\mu=\mu_n$ converges to u.

Concerning (1.1.2) we prove an existence result of solutions with a given trace belonging to the class of general outer regular Borel measures (not necessarily locally bounded).

Theorem 1.1.3 *Assume $1<q<q_c$ and $\mathcal{S}\subsetneq\partial\Omega$ is closed and μ is a positive Radon measure on $\mathcal{R}:=\partial\Omega\backslash\mathcal{S}$, then there exists a positive solution u of* (1.1.2) *such that $tr_{\partial\Omega}(u)=(\mathcal{S},\mu)$.*

When $1<q<q_c$ we prove a stronger result, using the characterization of singular solutions with strong singularities (see Theorem 1.1.6 below). When $q_c\leq q<2$ we prove that Theorem 1.1.3 still holds if $\mu=0$ and $\mathcal{S}=\overline{G}$ where $G\subsetneq\partial\Omega$ is relatively open, ∂G satisfies an interior sphere condition. Surprisingly the condition $\mathcal{S}\subsetneq\partial\Omega$ is necessary since there cannot exists any *large solution*, i.e. a solution which blows-up everywhere on $\partial\Omega$.

In order to characterize isolated singularities of positive solutions of (1.1.2) we introduce the following problem on the upper hemisphere S_+^{N-1} of the unit sphere in $\mathbb{R}^N$

$$\begin{cases} -\Delta'\omega+\left(\left(\frac{2-q}{q-1}\right)^2\omega^2+|\nabla'\omega|^2\right)^{\frac{q}{2}}-\frac{2-q}{q-1}\left(\frac{q}{q-1}-N\right)\omega=0 & \text{in } S_+^{N-1}\\ \omega=0 & \text{on } \partial S_+^{N-1}, \end{cases} \tag{1.1.14}$$

where ∇' and Δ' denote respectively the covariant gradient and the Laplace-Beltrami operator on S^{N-1}. To any solution ω of (1.1.14) we can associate a singular separable solution u_s of (1.1.2) in $\mathbb{R}_+^N:=\{x=(x_1,x_2,...,x_N)=(x',x_N):x_N>0\}$ vanishing on $\partial\mathbb{R}_+^N\setminus\{0\}$ written in spherical coordinates $(r,\sigma)=(|x|,\frac{x}{|x|})$

$$u_s(x)=u_s(r,\sigma)=r^{-\frac{2-q}{q-1}}\omega(\sigma)\qquad\forall x\in\overline{\mathbb{R}_+^N}\setminus\{0\}. \tag{1.1.15}$$

Theorem 1.1.4 *The problem* (1.1.14) *admits a positive solution if and only if* $1 < q < q_c$. *Furthermore this solution is unique and denoted by* ω_s.

This singular solution plays a fundamental role for describing isolated singularities.

Theorem 1.1.5 *Assume* $1 < q < q_c$ *and* $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ *is a nonnegative solution of* (1.1.2) *which vanishes on* $\partial\Omega \setminus \{0\}$. *Then the following dichotomy occurs :*

(i) Either there exists $c \geq 0$ *such that* $u = u_{c\delta_0}$ *solves* (1.1.3) *with* $g(r) = r^q$, $\mu = c\delta_0$ *and*

$$u(x) = cP^{\Omega}(x,0)(1 + o(1)) \qquad \text{as } x \to 0 \tag{1.1.16}$$

where P^{Ω} *is the Poisson kernel in* Ω.

(ii) Or $u = \lim_{c\to\infty} u_{c\delta_0}$ *and*

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} = \sigma \in S_+^{N-1}}} |x|^{\frac{2-q}{q-1}} u(x) = \omega_s(\sigma). \tag{1.1.17}$$

We also give a sharp estimate from below for singular points of the trace

Theorem 1.1.6 *Assume* $1 < q < q_c$ *and* u *is a positive solution of* (1.1.2) *with boundary trace* $(\mathcal{S}(u), \mu)$. *Then for any* $z \in \mathcal{S}(u)$ *there holds*

$$u(x) \geq u_{\infty\delta_z}(x) := \lim_{c\to\infty} u_{c\delta_z}(x) \qquad \forall x \in \Omega. \tag{1.1.18}$$

The description of $u_{\infty\delta_z}$ *is provided by* u_s *defined in* (1.1.15), *up to a translation and a rotation.*

The critical exponent q_c plays for removability of isolated boundary singularities (1.1.2) a similar role than q_s plays for (1.1.11) since we prove

Theorem 1.1.7 *Assume* $q_c \leq q < 2$, *then any nonnegative solution* $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ *of* (1.1.2) *vanishing on* $\partial\Omega \setminus \{0\}$ *is identically zero.*

The supercritical case for equation (1.1.2) can be understood using the Bessel capacity $C_{\frac{2-q}{q},q'}$ in dimension $N-1$, however we can only deal with moderate and sigma-moderate solutions. Following Dynkin [11], [14] we define

Definition 1.1.8 *A positive solution* u *of* (1.1.2) *is moderate if there exists a measure* $\mu \in \mathfrak{M}_+(\partial\Omega)$ *such that* u *solves problem* (1.1.3) *with* $g(r) = r^q$. *It is sigma-moderate if there exists an increasing sequence* $\{\mu_n\} \subset \mathfrak{M}_+(\partial\Omega)$ *such that the sequence of solutions* $\{u_{\mu_n}\}$ *increases and converges to* u *when* $n \to \infty$, *locally uniformly in* Ω.

Equivalently we shall prove that a positive solution u is moderate if and only if it is integrable in Ω and $|\nabla u| \in L^q_d(\Omega)$.

Theorem 1.1.9 *Assume $q_c \leq q < 2$ and $K \subset \partial\Omega$ is compact and satisfies $C_{\frac{2-q}{q},q'}(K) = 0$. Then any positive moderate solution $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus K)$ of* (1.1.2) *vanishing on $\partial\Omega \setminus K$ is identically zero.*

As a corollary we prove that the above result remains true if u is a sigma-moderate solution of (1.1.2). The counterpart of this result is the following necessary condition for solving problem (1.1.3).

Theorem 1.1.10 *Assume $q_c \leq q < 2$ and u is a positive moderate solution of* (1.1.2) *with boundary data $\mu \in \mathfrak{M}_+(\partial\Omega)$. Then μ is absolutely continuous with respect to the $C_{\frac{2-q}{q},q'}$-capacity.*

We end this chapter with some result concerning question of existence and removability of solutions of

$$-\Delta u + \tilde{g}(|\nabla u|) = \tilde{\mu} \qquad \text{in } \Omega \tag{1.1.19}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $\tilde{\mu}$ is a positive bounded Radon measure on Ω. We prove that if $\tilde{g}$ is a locally Lipschitz nondecreasing function vanishing at 0 and such that

$$\int_1^\infty \tilde{g}(s) s^{-\frac{2N-1}{N-1}} ds < \infty \tag{1.1.20}$$

then problem (1.1.19) admits a solution. In the power case

$$-\Delta u + |\nabla u|^q = \tilde{\mu} \qquad \text{in } \Omega \tag{1.1.21}$$

with $1 < q < 2$, the critical exponent is $q^* = \frac{N}{N-1}$. We prove that a necessary condition for solving (1.1.21) with a positive Radon measure $\tilde{\mu}$ is that $\tilde{\mu}$ vanishes on Borel subsets E with $C_{1,q'}$-capacity zero. The associated removability statement asserts that if K is a compact subset of Ω such that $C_{1,q'}(K) = 0$, any positive solution of

$$-\Delta u + |\nabla u|^q = 0 \qquad \text{in } \Omega \setminus K \tag{1.1.22}$$

satisfying that $\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS$ is bounded can be extended as a solution in whole Ω.

1.2 The Dirichlet problem and the boundary trace

Throughout this article Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with a C^2 boundary $\partial\Omega$ and c will denote a positive constant, independent of the data, the value of which may change from line to line. When needed the constant will be denoted by c_i or C_i for some indices $i = 1, 2, ...$, or some dependence will be made explicit such as $c_i(a, b, ...)$ or $C_i(a, b, ...)$ for some data a, b...

1.2.1 Boundary data bounded measures

We consider the following problem where μ belongs to the set $\mathfrak{M}^b(\partial\Omega)$

$$\begin{cases} -\Delta u + g(|\nabla u|) = 0 & \text{in } \Omega \\ u = \mu & \text{on } \partial\Omega. \end{cases} \tag{1.2.1}$$

We assume that g belongs to the class $\mathcal{G}_0$ which means that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lipschitz continuous nonnegative and nondecreasing function vanishing at 0. The *integral subcriticality condition* is the following

$$\int_1^\infty g(s)s^{-\frac{2N+1}{N}}\,ds < \infty. \tag{1.2.2}$$

If $g(r) = r^q$ the integral subcriticality condition is satisfied if $0 < q < q_c := \frac{N+1}{N}$.

Definition 1.2.1 *A function $u \in L^1(\Omega)$ such that $g(|\nabla u|) \in L^1_d(\Omega)$ is a weak solution of* (1.2.1) *if*

$$\int_\Omega (-u\Delta\zeta + g(|\nabla u|)\zeta)\,dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu \tag{1.2.3}$$

for all $\zeta \in X(\Omega) := \{\phi \in C^1_0(\overline{\Omega}) : \Delta\phi \in L^\infty(\Omega)\}$.

If we denote respectively by G^Ω and P^Ω the Green kernel and the Poisson kernel in Ω, with corresponding operators $\mathbb{G}^\Omega$ and $\mathbb{P}^\Omega$ it is classical from linear theory that the above definition is equivalent to

$$u = \mathbb{P}^\Omega[\mu] - \mathbb{G}^\Omega[g(|\nabla u|)]. \tag{1.2.4}$$

We recall that $M^p_h(\Omega)$ denote the Marcinkiewicz space (or weak L^p space) of exponent $p \geq 1$ and weight $h > 0$ defined by

$$M^p_h(\Omega) = \left\{ v \in L^1_{loc}(\Omega) : \exists C \geq 0 \text{ s. t. } \int_E |v|h dx \leq C|E|_h^{1-\frac{1}{p}}, \forall E \subset \Omega, E \text{ Borel} \right\}, \tag{1.2.5}$$

where $|E|_h = \int \chi_E h dx$. The smallest constant C for which (1.2.5) holds is the Marcinkiewicz norm of v denoted by $\|v\|_{M^p_h(\Omega)}$ and the following inequality will be much useful :

$$|\{x : |v(x)| \geq \lambda\}|_h \leq \lambda^{-p} \|v\|^p_{M^p_h(\Omega)} \qquad \forall \lambda > 0. \tag{1.2.6}$$

The main result of this section is the following existence and stability result for problem (1.2.1).

Theorem 1.2.2 *Assume $g \in \mathcal{G}_0$ satisfies* (1.2.2)*, then for any measure $\mu \in \mathfrak{M}_+(\partial\Omega)$ there exists a maximal solution $\bar{u} = \overline{u}_\mu$ to problem* (1.2.1)*. Furthermore $\overline{u} \in M^{\frac{N}{N-1}}(\Omega)$ and $|\nabla\overline{u}| \in M_d^{\frac{N+1}{N}}(\Omega)$. Finally, if $\{\mu_n\}$ is a sequence of positive bounded measures on $\partial\Omega$ which converges to μ in the weak sense of measures and $\{u_{\mu_n}\}$ is a sequence of solutions of* (1.2.1) *with boundary data μ_n, then there exists a subsequence $\{u_{\mu_{n_k}}\}$ converging to a solution u_μ of* (1.2.1) *in $L^1(\Omega)$ and $\{g(|\nabla u_{\mu_{n_k}}|)\}$ converges to $g(|\nabla u_\mu|)$ in $L^1_d(\Omega)$.*

We recall the following estimates [8], [16], [40] and [41].

Proposition 1.2.3 *For any $\alpha \in [0,1]$, there exists a positive constant c_1 depending on α, Ω and N such that*

$$\left\|\mathbb{G}^{\Omega}[\nu]\right\|_{L^1(\Omega)} + \left\|\mathbb{G}^{\Omega}[\nu]\right\|_{M_{d^\alpha}^{\frac{N+\alpha}{N+\alpha-2}}(\Omega)} \leq c_1 \left\|\nu\right\|_{\mathfrak{M}_{d^\alpha}(\Omega)}, \tag{1.2.7}$$

$$\left\|\nabla\mathbb{G}^{\Omega}[\nu]\right\|_{M_{d^\alpha}^{\frac{N+\alpha}{N+\alpha-1}}(\Omega)} \leq c_1 \left\|\nu\right\|_{\mathfrak{M}_{d^\alpha}(\Omega)}, \tag{1.2.8}$$

where

$$\left\|\nu\right\|_{\mathfrak{M}_{d^\alpha}(\Omega)} := \int_\Omega d^\alpha(x) d|\nu| \qquad \forall \nu \in \mathfrak{M}_{d^\alpha}(\Omega), \tag{1.2.9}$$

$$\left\|\mathbb{P}^{\Omega}[\mu]\right\|_{L^1(\Omega)} + \left\|\mathbb{P}^{\Omega}[f]\right\|_{M^{\frac{N}{N-1}}(\Omega)} + \left\|\mathbb{P}^{\Omega}[\mu]\right\|_{M_d^{\frac{N+1}{N-1}}(\Omega)} \leq c_1 \left\|\mu\right\|_{\mathfrak{M}(\partial\Omega)}, \tag{1.2.10}$$

$$\left\|\nabla\mathbb{P}^{\Omega}[\mu]\right\|_{M_d^{\frac{N+1}{N}}(\Omega)} \leq c_1 \left\|\mu\right\|_{\mathfrak{M}(\partial\Omega)}, \tag{1.2.11}$$

for any $\nu \in L^1_{d^\alpha}(\Omega)$ and any $\mu \in \mathfrak{M}(\partial\Omega)$.

Since $\partial\Omega$ is C^2, there exists $\delta^* > 0$ such that for any $\delta \in (0, \delta^*]$ and $x \in \Omega$ such that $d(x) < \delta$, there exists a unique $\sigma(x) \in \partial\Omega$ satisfying $|x - \sigma(x)| = d(x)$. We set $\sigma(x) = Proj_{\partial\Omega}(x)$. Furthermore, if $\mathbf{n} = \mathbf{n}_{\sigma(x)}$ is the normal outward unit vector to $\partial\Omega$ at $\sigma(x)$, we have $x = \sigma(x) - d(x)\mathbf{n}_{\sigma(x)}$. For $\delta \in (0, \delta^*]$, we set

$$\begin{aligned}
&\Omega_\delta = \{x \in \Omega : d(x) < \delta\},\\
&\Omega'_\delta = \{x \in \Omega : d(x) > \delta\},\\
&\Sigma_\delta = \partial\Omega'_\delta = \{x \in \Omega : d(x) = \delta\},\\
&\Sigma := \Sigma_0 = \partial\Omega.
\end{aligned}$$

For any $\delta \in (0, \delta^*)$, the mapping $x \mapsto (d(x), \sigma(x))$ defines a C^1 diffeomorphism from Ω_δ to $(0, \delta) \times \Sigma$. Therefore we can write $x = \sigma(x) - d(x)\mathbf{n}_{\sigma(x)}$ for every $x \in \Omega_\delta$. Any point $x \in \overline{\Omega}_{\delta^*}$ is represented by a unique couple $(\delta, \sigma) \in [0, \delta^*] \times \Sigma$ with formula $x = \sigma - \delta\mathbf{n}_\sigma$. This system of coordinates which will be made more precise in the boundary trace construction is called *flow coordinates.*

Proof of Theorem 1.2.2. *Step 1 : Construction of approximate solutions.* Let $\{\mu_n\}$ be a sequence of positive functions in $C^1(\partial\Omega)$ such that $\{\mu_n\}$ converges to μ in the weak sense of measures and $\|\mu_n\|_{L^1(\partial\Omega)} \leq c_2 \|\mu\|_{\mathfrak{M}(\partial\Omega)}$ for all n, where c_2 is a positive constant independent of n. We next consider the following problem

$$\left\{\begin{aligned} -\Delta v + g(|\nabla(v + \mathbb{P}^{\Omega}[\mu_n])|) &= 0 \quad \text{in } \Omega\\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right. \tag{1.2.12}$$

It is easy to see that 0 and $-\mathbb{P}^{\Omega}[\mu_n]$ are respectively supersolution and subsolution of (1.2.12). By [26, Theorem 6.5] there exists a solution $v_n \in W^{2,p}(\Omega)$ with $1 < p < \infty$ to problem (1.2.12) satisfying $-\mathbb{P}^{\Omega}[\mu_n] \leq v_n \leq 0$. Thus the function $u_n = v_n + \mathbb{P}^{\Omega}[\mu_n]$ is a solution of

$$\left\{ \begin{array}{ll} -\Delta u_n + g(|\nabla u_n|) = 0 & \text{in } \Omega \\ u_n = \mu_n & \text{on } \partial\Omega. \end{array} \right. \tag{1.2.13}$$

By the maximum principle, such solution is the unique solution of (1.2.13).

Step 2 : We claim that the sequences $\{u_n\}$ and $\{|\nabla u_n|\}$ remain uniformly bounded respectively in $M^{\frac{N}{N-1}}(\Omega)$ and $M_d^{\frac{N+1}{N}}(\Omega)$. Let ξ be the solution to

$$\left\{ \begin{array}{ll} -\Delta \xi = 1 & \text{in } \Omega \\ \xi = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.2.14}$$

then there exists a constant $c_3 > 0$ such that

$$\frac{1}{c_3} < -\frac{\partial \xi}{\partial \mathbf{n}} < c_3 \text{ and } \frac{d(x)}{c_3} \leq \xi \leq c_3 d(x). \tag{1.2.15}$$

By multiplying the equation in (1.2.13) by ξ and integrating on Ω, we obtain

$$\int_\Omega u_n dx + \int_\Omega g(|\nabla u_n|)\xi dx = -\int_{\partial\Omega} \mu_n \frac{\partial \xi}{\partial \mathbf{n}} dS,$$

which implies

$$\int_\Omega u_n dx + \int_\Omega d(x) g(|\nabla u_n|) dx \leq c_4 \|\mu\|_{\mathfrak{M}(\partial\Omega)} \tag{1.2.16}$$

where c_4 is a positive constant independent of n. By Proposition 1.2.3 and by noticing that $u_n \leq \mathbb{P}^{\Omega}[\mu_n]$, we get

$$\|u_n\|_{M^{\frac{N}{N-1}}(\Omega)} \leq \left\|\mathbb{P}^{\Omega}[\mu_n]\right\|_{M^{\frac{N}{N-1}}(\Omega)} \leq c_1 \|\mu_n\|_{L^1(\partial\Omega)} \leq c_1 c_2 \|\mu\|_{\mathfrak{M}(\partial\Omega)}. \tag{1.2.17}$$

Set $f_n = -g(|\nabla u_n|)$ then $f_n \in L^1_d(\Omega)$ and u_n satisfies

$$\int_\Omega (-u_n \Delta\zeta - f_n \zeta) dx = -\int_{\partial\Omega} \mu_n \frac{\partial \zeta}{\partial \mathbf{n}} dS \tag{1.2.18}$$

for any $\zeta \in X(\Omega)$. From (1.2.4) and Proposition 1.2.3, we derive that

$$\|\nabla u_n\|_{M_d^{\frac{N+1}{N}}(\Omega)} \leq c_1 \left(\|f_n\|_{L^1_d(\Omega)} + \|\mu_n\|_{L^1(\partial\Omega)} \right), \tag{1.2.19}$$

which, along with (1.2.16), implies that

$$\|\nabla u_n\|_{M_d^{\frac{N+1}{N}}(\Omega)} \leq c_5 \|\mu\|_{\mathfrak{M}(\partial\Omega)} \tag{1.2.20}$$

where c_5 is a positive constant depending only on Ω and N. Thus the claim follows from (1.2.17) et (1.2.20).

Step 3 : Existence of a solution. By standard results on elliptic equations and measure theory [9, Cor. IV 27], the sequences $\{u_n\}$ and $\{|\nabla u_n|\}$ are compact in $L^1_{loc}(\Omega)$. Therefore, there exist a subsequence, still denoted by $\{u_n\}$, and a function u such that $\{u_n\}$ converges to u in $L^1_{loc}(\Omega)$ and a.e. in Ω.

(i) The sequence $\{u_n\}$ converges to u in $L^1(\Omega)$: let $E \subset \Omega$ be a Borel subset, then

$$\int_E u_n dx \leq |E|^{\frac{1}{N}} \|u_n\|_{M^{\frac{N}{N-1}}(\Omega)} \leq c_1 c_2 |E|^{\frac{1}{N}} \|\mu\|_{\mathfrak{M}(\partial\Omega)} . \tag{1.2.21}$$

The compactness follows by Vitali's theorem.

(ii) The sequence $g(|\nabla u_n|)$ converges to $g(|\nabla u|)$ in $L^1_d(\Omega)$: consider again a Borel set $E \subset \Omega$, $\lambda > 0$ and write

$$\int_E d(x)g(|\nabla u_n|)dx \leq \int_{E\cap\{x:|\nabla u_n(x)|\leq\lambda\}} d(x)g(|\nabla u_n|)dx + \int_{\{x:|\nabla u_n(x)|>\lambda\}} d(x)g(|\nabla u_n|)dx.$$

First

$$\int_{E\cap\{x:|\nabla u_n(x)|\leq\lambda\}} d(x)g(|\nabla u_n|)dx \leq g(\lambda)|E|_d. \tag{1.2.22}$$

Then

$$\int_{E\cap\{x:|\nabla u_n(x)|>\lambda\}} d(x)g(|\nabla u_n|)dx \leq -\int_\lambda^\infty g(s)d\omega_n(s)$$

where $\omega_n(s) = |\{x \in \Omega : |\nabla u_n(x)| > s\}|_d$. Using the fact that $g' \geq 0$ combined with (1.2.6) and (1.2.20), we get

$$\begin{aligned}
-\int_\lambda^t g(s)d\omega_n(s) &= g(\lambda)\omega_n(\lambda) - g(t)\omega_n(t) + \int_\lambda^t \omega_n(s)g'(s)ds \\
&\leq g(\lambda)\omega_n(\lambda) - g(t)\omega_n(t) + c_6 \|\mu\|^{\frac{N+1}{N}}_{\mathfrak{M}(\partial\Omega)} \int_\lambda^t s^{-\frac{N+1}{N}} g'(s)ds \\
&\leq \left(\omega_n(\lambda) - c_6 \|\mu\|^{\frac{N+1}{N}}_{\mathfrak{M}(\partial\Omega)} \lambda^{-\frac{N+1}{N}}\right) g(\lambda) - \left(\omega_n(t) - c_6 \|\mu\|^{\frac{N+1}{N}}_{\mathfrak{M}(\partial\Omega)} t^{-\frac{N+1}{N}}\right) g(t) \\
&\qquad + c_6 \frac{N+1}{N} \|\mu\|^{\frac{N+1}{N}}_{\mathfrak{M}(\partial\Omega)} \int_\lambda^t g(s)s^{-\frac{2N+1}{N}} ds.
\end{aligned}$$

We have already used the fact that $\omega_n(\lambda) \leq c_6 \|\mu\|^{\frac{N+1}{N}}_{\mathfrak{M}(\partial\Omega)} \lambda^{-\frac{N+1}{N}}$, and since the condition (1.2.2) holds, $\liminf_{t\to\infty} t^{-\frac{N+1}{N}} g(t) = 0$. Letting $t \to \infty$ we derive

$$\int_{E\cap\{x:|\nabla u_n(x)|>\lambda\}} d(x)g(|\nabla u_n|)dx \leq c_6 \frac{N+1}{N} \|\mu\|^{\frac{N+1}{N}}_{\mathfrak{M}(\partial\Omega)} \int_\lambda^\infty g(s)s^{-\frac{2N+1}{N}} ds. \tag{1.2.23}$$

For $\epsilon > 0$ we fix λ in order the right-hand side of (1.2.23) be smaller than $\frac{\epsilon}{2}$. Thus, if $|E|_d \leq \frac{\epsilon}{2g(\lambda)+1}$, we obtain

$$\int_E d(x)g(|\nabla u_n|)dx \leq \epsilon. \tag{1.2.24}$$

The convergence follows again by Vitali's theorem. Next for any $\zeta \in X(\Omega)$, we have

$$\int_\Omega (-u_n\Delta\zeta + g(|\nabla u_n|)\zeta)dx = -\int_{\partial\Omega} \mu_n \frac{\partial\zeta}{\partial\mathbf{n}} dS \tag{1.2.25}$$

By taking into account the fact that $|\zeta| \leq cd$ in Ω, we can pass to the limit in each term in (1.2.25) and obtain (1.2.3); so u is a solution of (1.2.1). Clearly $u \in M^{\frac{N}{N-1}}(\Omega)$ and $|\nabla u| \in M_d^{\frac{N+1}{N}}(\Omega)$ from (1.2.4) and Proposition 1.2.3.

Step 4 : Existence of a maximal solution. We first notice that any solution u of (1.2.1) is smaller than $\mathbb{P}^{\Omega}[\mu]$. Then $u \leq \mathbb{P}^{\Omega}[\mu]$ in Ω'_δ and by the maximum principle $u \leq u_\delta$ which satisfies

$$\begin{cases} -\Delta u_\delta + g(|\nabla u_\delta|) = 0 & \text{in } \Omega'_\delta \\ u_\delta = \mathbb{P}^{\Omega}[\mu] & \text{on } \Sigma_\delta. \end{cases} \tag{1.2.26}$$

As a consequence, $0 < \delta < \delta' \Longrightarrow u_\delta \leq u_{\delta'}$ in $\Omega'_{\delta'}$ and $u_\delta \downarrow \bar{u}_\mu$ which is not zero if μ is so, since it is bounded from below by the already constructed solution u. We extend u_δ, $|\nabla u_\delta|$ and $g(|\nabla u_\delta|)$ by zero outside $\overline{\Omega}'_\delta$ and still denote them by the same expressions. Let $E \subset \Omega$ be a Borel set and put $E_\delta = E \cap \Omega'_\delta$ then (1.2.21) becomes

$$\begin{aligned} \int_{E_\delta} u_\delta dx \leq |E_\delta|^{\frac{1}{N}} \|u_\delta\|_{M^{\frac{N}{N-1}}(\Omega'_\delta)} &\leq c_1 c_2 |E_\delta|^{\frac{1}{N}} \left\| \mathbb{P}^{\Omega}[\mu]|_{\Sigma_\delta} \right\|_{\mathfrak{M}(\Sigma_\delta)} \\ &\leq c_1 c_2 c_7 |E|^{\frac{1}{N}} \|\mu\|_{\mathfrak{M}(\Sigma)}. \end{aligned} \tag{1.2.27}$$

Set $d_\delta(x) := \text{dist}\,(x, \Omega_\delta)$ $(= (d(x) - \delta)_+$ if $x \in \Omega_{\delta^*})$, we have

$$\int_{E_\delta \cap \{x : |\nabla u_\delta| > \lambda\}} d_\delta(x) g(|\nabla u_\delta|) dx \leq -\int_\lambda^\infty g(s) d\omega_\delta(s),$$

where $\omega_\delta(s) = |\{x \in \Omega : |\nabla u_\delta(x)| > s\}|_{d_\delta}$. Since $\left\| \mathbb{P}^{\Omega}[\mu]|_{\Sigma_\delta} \right\|_{\mathfrak{M}(\Sigma_\delta)} \leq c_7 \|\mu\|_{\mathfrak{M}(\Sigma)}$, (1.2.22) and (1.2.23) become respectively

$$\int_{E_\delta \cap \{x : |\nabla u_\delta(x)| \leq \lambda\}} d_\delta(x) g(|\nabla u_\delta|) dx \leq g(\lambda) |E_\delta|_{d_\delta}. \tag{1.2.28}$$

and

$$\int_{E_\delta \cap \{x : |\nabla u_\delta(x)| > \lambda\}} d_\delta(x) g(|\nabla u_\delta|) dx \leq c_6 \frac{N+1}{N} \|\mu\|_{\mathfrak{M}}^{\frac{N+1}{N}} \int_\lambda^\infty g(s) s^{-\frac{2N+1}{N}} ds. \tag{1.2.29}$$

Combining (1.2.28) and (1.2.29) and noting that $|E_\delta|_{d_\delta} \leq |E|_d$, we obtain that for any $\epsilon > 0$ there exists $\lambda > 0$, independent of δ by (1.2.28), such that

$$\int_{E_\delta} d_\delta(x) g(|\nabla u_\delta|) dx \leq \epsilon. \tag{1.2.30}$$

provided $|E|_d \leq \frac{\epsilon}{2g(\lambda)+1}$.

Finally, if $\zeta \in X(\Omega)$ we denote by ζ_δ the solution of

$$\begin{cases} -\Delta \zeta_\delta = -\Delta \zeta & \text{in } \Omega'_\delta \\ \zeta_\delta = 0 & \text{on } \Sigma_\delta. \end{cases} \tag{1.2.31}$$

Then

$$\int_{\Omega'_\delta}(-u_\delta\Delta\zeta_\delta+g(|\nabla u_\delta|)\zeta_\delta)dx=-\int_{\Sigma_\delta}\frac{\partial\zeta_\delta}{\partial\mathbf{n}}\mathbb{P}^\Omega[\mu]dS \tag{1.2.32}$$

Clearly $|\zeta_\delta|\leq Cd_\delta$ and $\zeta_\delta\chi_{\Omega'_\delta}\to\zeta$ uniformly in Ω by standard elliptic estimates. Since the right-hand side of (1.2.32) converges to $-\int_{\partial\Omega}\frac{\partial\zeta}{\partial\mathbf{n}}d\mu$, it follows by Vitali's theorem that $\bar{u}_\mu$ satisfies (1.2.3).

Step 5 : Stability. Consider a sequence of positive bounded measures $\{\mu_n\}$ which converges weakly to μ. By estimates (1.2.17) and (1.2.20), $\{u_{\mu_n}\}$ and $\{g(|\nabla u_{\mu_n}|)\}$ are relatively compact in $L^1_{loc}(\Omega)$ and respectively uniformly integrable in $L^1(\Omega)$ and $L^1_d(\Omega)$. Up to a subsequence, they converge a.e. respectively to u and $g(|\nabla u|)$ for some function u. As in Step 3, u is a solution of (1.2.1). □

A variant of the stability statement is the following result which will be much useful in the analysis of the boundary trace. The proof is similar as Step 4 in the proof of Theorem 1.2.2.

Corollary 1.2.4 *Let g in $\mathcal{G}_0$ satisfy* (1.2.2). *Assume $\{\delta_n\}$ is a sequence decreasing to* 0 *and $\{\mu_n\}$ is a sequence of positive bounded measures on $\Sigma_{\delta_n}=\partial\Omega'_{\delta_n}$ which converges to μ in the weak sense of measures and let u_{μ_n} be solutions of* (1.2.1) *with boundary data μ_n. Then there exists a subsequence $\{u_{\mu_{n_k}}\}$ of solutions of* (1.2.1) *with boundary data μ_{n_k} which converges to a solution u_μ with boundary data μ.*

1.2.2 Boundary trace

The construction of the boundary trace of positive solutions of (1.1.1) is a combination of tools developed in [29]–[31] with the help of a geometric construction from [3].

Definition 1.2.5 *Let $\mu_\delta\in\mathfrak{M}(\Sigma_\delta)$ for all $\delta\in(0,\delta^*)$ and $\mu\in\mathfrak{M}(\Sigma)$. We say that $\mu_\delta\to\mu$ as $\delta\to0$ in the sense of weak convergence of measures if*

$$\lim_{\delta\to0}\int_{\Sigma_\delta}\phi(\sigma(x))d\mu_\delta=\int_\Sigma\phi d\mu\qquad\forall\phi\in C_c(\Sigma). \tag{1.2.33}$$

A function $u\in C(\Omega)$ possesses a measure boundary trace $\mu\in\mathfrak{M}(\Sigma)$ if

$$\lim_{\delta\to0}\int_{\Sigma_\delta}\phi(\sigma(x))u(x)dS=\int_\Sigma\phi d\mu\qquad\forall\phi\in C_c(\Sigma). \tag{1.2.34}$$

Similarly, if A is a relatively open subset of Σ, we say that u possesses a trace μ on A in the sense of weak convergence of measures if $\mu\in\mathfrak{M}(A)$ and (1.2.34) *holds for every $\phi\in C_c(A)$.*

We recall the following result [32, Cor 2.3], adapted here to (1.1.1),

Proposition 1.2.6 *Assume $g : \mathbb{R}_+ \to \mathbb{R}_+$ and let $u \in C^2(\Omega)$ be a positive solution of* (1.1.1). *Suppose that for some $z \in \partial\Omega$ there exists an open neighborhood U such that*

$$\int_{U\cap\Omega} g(|\nabla u|)d(x)dx < \infty. \tag{1.2.35}$$

Then $u \in L^1(K \cap \Omega)$ for every compact set $K \subset U$ and there exists a positive Radon measure ν on $\Sigma \cap U$ such that

$$\lim_{\delta\to 0}\int_{\Sigma_\delta\cap U} \phi(\sigma(x))u(x)dS = \int_{\Sigma\cap U}\phi d\nu \qquad \forall\phi \in C_c(\Sigma\cap U). \tag{1.2.36}$$

Definition 1.2.7 *Let $u \in C^2(\Omega)$ be a positive solution of* (1.1.1). *A point $z \in \partial\Omega$ is a regular boundary point of u if there exists an open neighborhood U of z such that* (1.2.35) *holds. The set of regular points is denoted by $\mathcal{R}(u)$. Its complement $\mathcal{S}(u) = \partial\Omega \setminus \mathcal{R}(u)$ is called the singular boundary set of u.*

Clearly $\mathcal{R}(u)$ is relatively open and there exists a positive Radon measure μ on $\mathcal{R}(u)$ such that u admits $\mu := \mu(u)$ as a measure boundary trace on $\mathcal{R}(u)$ and $\mu(u)$ is uniquely determined. The couple $(\mathcal{S}(u), \mu)$ is called the *boundary trace* of u and denoted by $tr_{\partial\Omega}(u)$.

The main question is to determine the behaviour of u near $\mathcal{S}(u)$. The following result is proved in [32, Lemma 2.8].

Proposition 1.2.8 *Assume $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $u \in C^2(\Omega)$ be a positive solution of* (1.1.1) *with the singular boundary set $\mathcal{S}(u)$. If $z \in \mathcal{S}(u)$ is such that there exists an open neighborhood U' of z such that $u \in L^1(U' \cap \Omega)$, then for every neighborhood U of z there holds*

$$\lim_{\delta\to 0}\int_{\Sigma_\delta\cap U} u(x)dS = \infty. \tag{1.2.37}$$

Corollary 1.2.9 *Let $u \in C^2(\Omega)$ is a positive solution of* (1.1.2) *with $\frac{3}{2} < q \leq 2$. Then* (1.2.37) *holds for every $z \in \mathcal{S}(u)$.*

Proof. This is a direct consequence of Lemma 1.3.2 since $\frac{q-2}{q-1} > -1$ implies $u \in L^1(\Omega)$. □

We prove below that this result holds for any $1 < q \leq 2$.

Theorem 1.2.10 *Assume $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies*

$$\liminf_{r\to\infty}\frac{g(r)}{r^q} > 0 \tag{1.2.38}$$

where $1 < q \leq 2$. If $u \in C^2(\Omega)$ is a positive solution of (1.1.1), *then* (1.2.37) *holds for every $z \in \mathcal{S}(u)$.*

Proof. Up to rescaling we can assume that $g(r) \geq r^q - \tau$ for some $\tau \geq 0$. We recall some results from [6] in the form exposed in [3, Sect 2]. There exist an open cover $\{\Sigma_j\}_{j=1}^k$ of Σ, an open set $\mathcal{D}$ of $\mathbb{R}^{N-1}$ and C^2 mappings T_j from $\mathcal{D}$ to Σ_j with rank $N-1$ such that for each $\sigma \in \Sigma_j$ there exists a unique $a \in \mathcal{D}$ with the property that $\sigma = T_j(a)$. The couples $\{\mathcal{D}, T_j^{-1}\}$ form a system of local charts of Σ. If we set $\Omega_j = \{x \in \Omega_{\delta^*} : \sigma(x) \in \Sigma_j\}$ then for any $j = 1, ..., k$ the mapping

$$\Pi_j : (\delta, a) \mapsto x = T_j(a) - \delta \mathbf{n}$$

where $\mathbf{n}$ is the outward unit normal vector to Σ at $T_j(a) = \sigma(x)$ is a C^2 diffeomorphism from $(0, \delta^*) \times \mathcal{D}$ to Ω_j. The Laplacian obtains the following expressions in terms of this system of flow coordinates provided the lines $\sigma_i = ct$ are the vector fields of the principal curvatures $\bar{\kappa}_i$ on Σ

$$\Delta = \Delta_\delta + \Delta_\sigma \tag{1.2.39}$$

where

$$\Delta_\delta = \frac{\partial^2}{\partial \delta^2} - (N-1)H \frac{\partial}{\partial \delta} \tag{1.2.40}$$

with $H = H(\delta, .) = \frac{1}{N-1} \sum_{i=1}^{N-1} \frac{\bar{\kappa}_i}{1-\delta\bar{\kappa}_i}$ being the mean curvature of Σ_δ and

$$\Delta_\sigma = \frac{1}{\sqrt{|g|}} \sum_{i=1}^{N-1} \frac{\partial}{\partial \sigma_i} \left(\frac{\sqrt{|g|}}{\bar{g}_{ii}(1 - \delta\bar{\kappa}_i + \kappa_{ii}\delta^2)} \frac{\partial}{\partial \sigma_i} \right). \tag{1.2.41}$$

In this expression, $\bar{g} = (\bar{g}_{ij})$ is the metric tensor on Σ and it is diagonal by the choice of coordinates and $|g| = \Pi_{i=1}^{N-1} \bar{g}_{ii}(1 - \delta\bar{\kappa}_i)^2$. In particular

$$|\nabla \xi|^2 = \sum_{i=1}^{N-1} \frac{\xi_{\sigma_i}^2}{\bar{g}_{ii}(1 - \delta\bar{\kappa}_i + \kappa_{ii}\delta^2)} + \xi_\delta^2 \tag{1.2.42}$$

and

$$\nabla \xi . \nabla \eta = \sum_{i=1}^{N-1} \frac{\xi_{\sigma_i}\eta_{\sigma_i}}{\bar{g}_{ii}(1 - \delta\bar{\kappa}_i + \kappa_{ii}\delta^2)} + \xi_\delta \eta_\delta = \nabla_\sigma \xi . \nabla_\sigma \eta + \xi_\delta \eta_\delta. \tag{1.2.43}$$

If $z \in \mathcal{S}(u)$ we can assume that $U_\Sigma := U \cap \Sigma$ is smooth and contained in a single chart Σ_j. Let ϕ be the first eigenfunction of Δ_σ in $W_0^{1,2}(U_\Sigma)$ normalized so that $\max_{U_\Sigma} \phi = 1$ and $\alpha > 1$ to be made precise later on. From

$$-\Delta_\delta u - \Delta_\sigma u + \frac{1}{2}(|\nabla u|^q - \tau) + \frac{1}{2} g(|\nabla u|) \leq 0$$

we obtain by multiplying by ϕ^α and integrating over U_Σ

$$\begin{aligned} -\frac{d^2}{d\delta^2} \int_{U_\Sigma} u\phi^\alpha dS + (N-1) \int_{U_\Sigma} \frac{\partial u}{\partial \delta} \phi^\alpha H dS + \alpha \int_{U_\Sigma} \phi^{\alpha-1} \nabla_\sigma u . \nabla_\sigma \phi \, dS \\ + \frac{1}{2} \int_{U_\Sigma} \phi^\alpha (|\nabla u|^q - \tau) dS + \frac{1}{2} \int_{U_\Sigma} \phi^\alpha g(|\nabla u|) dS \leq 0. \end{aligned} \tag{1.2.44}$$

Provided $\alpha > q' - 1$ we obtain by Hölder inequality

$$\begin{aligned}\left|\int_{U_\Sigma}\phi^{\alpha-1}\nabla_\sigma u.\nabla_\sigma\phi dS\right| &\leq \left(\int_{U_\Sigma}|\nabla u|^q\phi^\alpha dS\right)^{\frac{1}{q}}\left(\int_{U_\Sigma}|\nabla_\sigma\phi|^{q'}\phi^{\alpha-q'}dS\right)^{\frac{1}{q'}} \\ &\leq \epsilon\int_{U_\Sigma}|\nabla u|^q\phi^\alpha dS + \epsilon^{\frac{1}{1-q}}\int_{U_\Sigma}|\nabla_\sigma\phi|^{q'}\phi^{\alpha-q'}dS,\end{aligned} \tag{1.2.45}$$

and

$$\left|\int_{U_\Sigma}\frac{\partial u}{\partial\delta}\phi^\alpha H dS\right| \leq \epsilon\left\|H\right\|_{L^\infty}\int_{U_\Sigma}|\nabla u|^q\phi^\alpha dS + \epsilon^{\frac{1}{1-q}}\left\|H\right\|_{L^\infty}\int_{U_\Sigma}\phi^\alpha dS \tag{1.2.46}$$

with $\epsilon > 0$. We derive, with ϵ small enough,

$$\frac{d^2}{d\delta^2}\int_{U_\Sigma}u\phi^\alpha dS \geq \left(\frac{1}{2} - c_8\epsilon\right)\int_{U_\Sigma}|\nabla u|^q\phi^\alpha dS + \frac{1}{2}\int_{U_\Sigma}\phi^\alpha g(|\nabla u|)dS - c_8' \tag{1.2.47}$$

where $c_8 = c_8(q, H)$ and $c_8' = c_8'(N, q, H)$. Integrating (1.2.47) twice yields to

$$\int_{U_\Sigma}u(\delta,.)\phi^\alpha dS \geq \left(\frac{1}{2} - c_8\epsilon\right)\int_\delta^{\delta^*}\int_{U_\Sigma}|\nabla u|^q\phi^\alpha dS(\tau-\delta)d\tau + \frac{1}{2}\int_{U_\Sigma}\phi^\alpha g(|\nabla u|)dS - c_8''. \tag{1.2.48}$$

Since $z \in \mathcal{S}(u)$, the right-hand side of (1.2.48) tends monotically to ∞ as $\delta \to 0$, which implies that (1.2.37) holds. □

Remark. It is often usefull to consider the couple $(\mathcal{S}(u), \mu)$ defining the boundary trace of u as an outer regular Borel measure ν uniquely determined by

$$\nu(E) = \begin{cases}\mu(E) & \text{if } E \subset \mathcal{R}(u) \\ \infty & \text{if } E \cap \mathcal{S}(u) \neq \emptyset\end{cases} \tag{1.2.49}$$

for all Borel set $E \subset \partial\Omega$, and we will denote $tr_{\partial\Omega}(u) = \nu$.

The integral blow-up estimate (1.2.37) remains valid if the growth estimate (1.2.38) is replaced by (1.2.2).

Theorem 1.2.11 *Assume $g \in \mathcal{G}_0$ satisfies* (1.2.2). *If $u \in C^2(\Omega)$ is a positive solution of* (1.1.1), *then* (1.2.37) *holds for every $z \in \mathcal{S}(u)$.*

Proof. By translation we assume $z = 0 \in \mathcal{S}(u)$ and (1.2.37) does not hold. We proceed by contradiction, assuming that there exists an open neighborhood U of z such that

$$\liminf_{\delta\to 0}\int_{\Sigma_\delta\cap U}udS < \infty. \tag{1.2.50}$$

By Proposition 1.2.8, for any neighborhood U' of z there holds

$$\int_{\Omega\cap U'}udx = \infty, \tag{1.2.51}$$

which implies

$$\limsup_{\delta\to 0}\int_{\Sigma_\delta\cap U'} udS=\infty. \tag{1.2.52}$$

For $n\in\mathbb{N}_*$, we take $U'=B_{\frac{1}{n}}$; there exists a sequence $\{\delta_{n,k}\}_{k\in\mathbb{N}}$ satisfying $\lim_{k\to\infty}\delta_{n,k}=0$ such that

$$\lim_{k\to\infty}\int_{\Sigma_{\delta_{n,k}}\cap B_{\frac{1}{n}}} udS=\infty. \tag{1.2.53}$$

Then, for any $\ell>0$, there exists $k_\ell:=k_{n,\ell}\in\mathbb{N}$ such that

$$k\geq k_\ell\Longrightarrow\int_{\Sigma_{\delta_{n,k}}\cap B_{\frac{1}{n}}} udS\geq\ell \tag{1.2.54}$$

and $k_{n,\ell}\to\infty$ when $n\to\infty$. In particular there exists $m:=m(\ell,n)>0$ such that

$$\int_{\Sigma_{\delta_{n,k_\ell}}\cap B_{\frac{1}{n}}}\inf\{u,m\}dS=\ell. \tag{1.2.55}$$

By the maximum principle u is bounded from below in $\Omega'_{\delta_{n,k_\ell}}$ by the solution $v:=v_{\delta_{n,k_\ell}}$ of

$$\left\{\begin{array}{ll} -\Delta v+g(|\nabla v|)=0 & \text{in } \Omega'_{\delta_{n,k_\ell}}\\ \qquad\qquad v=\inf\{u,m\} & \text{on } \Sigma_{\delta_{n,k_\ell}}.\end{array}\right. \tag{1.2.56}$$

When $n\to\infty$, $\inf\{u,m(\ell,n)\}dS$ converges in the weak sense of measures to $\ell\delta_0$. By Corollary 1.2.4 there exists a solution $u_{\ell\delta_0}$ such that $v_{\delta_{n,k_\ell}}\to u_{\ell\delta_0}$ when $n\to\infty$ and consequently $u\geq u_{\ell\delta_0}$ in Ω. Even if $u_{\ell\delta_0}$ may not be unique, this implies

$$\liminf_{\delta\to 0}\int_{\Sigma_\delta} u\zeta(x)dS\geq\lim_{\delta\to 0}\int_{\Sigma_\delta} u_{\ell\delta_0}\zeta(x)dS=\ell \tag{1.2.57}$$

for any nonnegative $\zeta\in C^\infty(\mathbb{R}^N)$ such that $\zeta=1$ in a neighborhood of 0. Since ℓ is arbitrary we obtain

$$\liminf_{\delta\to 0}\int_{\Sigma_\delta} u\zeta(x)dS=\infty \tag{1.2.58}$$

which contradicts (1.2.50). □

1.3 Boundary singularities

1.3.1 Boundary data unbounded measures

We recall that for any $q>1$, any solution u of (1.1.2) bounded from below satisfies [19, Theorem A1] the following estimate : for any $\epsilon>0$, there exists $C_\epsilon>0$ such that

$$\sup_{d(x)\geq\epsilon}|\nabla u(x)|\leq C_\epsilon. \tag{1.3.1}$$

Later on Lions gave in [24, Th IV 1] a more precise estimate that we recall below.

Lemma 1.3.1 *Assume $q > 1$ and $u \in C^2(\Omega)$ is any solution of* (1.1.2) *in* Ω. *Then*

$$|\nabla u(x)| \leq C_1(N,q)(d(x))^{-\frac{1}{q-1}} \qquad \forall x \in \Omega. \tag{1.3.2}$$

Similarly, the following result is proved in [24].

Lemma 1.3.2 *Assume $q > 1$ and $u \in C^2(\Omega)$ is a solution of* (1.1.2) *in* Ω. *Then*

$$|u(x)| \leq \frac{C_2(N,q)}{2-q}\left((d(x))^{\frac{q-2}{q-1}} - (\delta^*)^{\frac{q-2}{q-1}}\right) + \max\{|u(z)| : z \in \Sigma_{\delta^*}\} \qquad \forall x \in \Omega \tag{1.3.3}$$

if $q \neq 2$, and

$$|u(x)| \leq C_3(N)\left(\ln \delta^* - \ln d(x)\right) + \max\{|u(z)| : z \in \Sigma_{\delta^*}\} \qquad \forall x \in \Omega \tag{1.3.4}$$

if $q = 2$, for some $C_2(N,q), C_3(N) > 0$.

Proof. Put $M_{\delta^*} := \max\{|u(z)| : z \in \Sigma_{\delta^*}\}$ and let $x \in \Omega_{\delta^*}$, $x = \sigma(x) - d(x)\mathbf{n}_{\sigma(x)}$, and $x_0 = \sigma(x) - \delta^*\mathbf{n}_{\sigma(x)}$. Then, using Lemma 1.3.1, we get

$$\begin{aligned} |u(x)| &\leq M_{\delta^*} + \int_0^1 \left|\tfrac{d}{dt}u(tx + (1-t)x_0)\right| dt \\ &\leq M_{\delta^*} + C_1(N,q)\int_0^1 (td(x) + (1-t)\delta^*)^{-\frac{1}{q-1}}(\delta^* - d(x))dt. \end{aligned} \tag{1.3.5}$$

Thus we obtain (1.3.3) or (1.3.4) according the value of q. □

If $q = 2$ and u solves (1.1.2), $v = e^u$ is harmonic and positive while if $q > 2$, any solution remains bounded in Ω. Although this last case is interesting in itself, we will consider only the case $1 < q < 2$.

Lemma 1.3.3 *Assume $1 < q < 2$, $0 \in \partial\Omega$ and $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a solution of* (1.1.2) *in Ω which vanishes on $\partial\Omega \setminus \{0\}$. Then*

$$u(x) \leq C_4(q)|x|^{\frac{q-2}{q-1}} \qquad \forall x \in \Omega. \tag{1.3.6}$$

Proof. For $\epsilon > 0$, we set

$$P_\epsilon(r) = \begin{cases} 0 & \text{if } r \leq \epsilon \\ \frac{-r^4}{2\epsilon^3} + \frac{3r^3}{\epsilon^2} - \frac{6r^2}{\epsilon} + 5r - \frac{3\epsilon}{2} & \text{if } \epsilon < r < 2\epsilon \\ r - \frac{3\epsilon}{2} & \text{if } r \geq 2\epsilon \end{cases}$$

and let u_ϵ be the extension of $P_\epsilon(u)$ by zero outside Ω. There exists R_0 such that $\Omega \subset B_{R_0}$. Since $0 \leq P'_\epsilon(r) \leq 1$ and P_ϵ is convex, $u_\epsilon \in C^2(\mathbb{R}^N)$ and it satisfies $-\Delta u_\epsilon + |\nabla u_\epsilon|^q \leq 0$. Furthermore u_ϵ vanishes in $B^c_{R_0}$. For $R \geq R_0$ we set

$$U_{\epsilon,R}(x) = C_4(q)\left((|x| - \epsilon)^{\frac{q-2}{q-1}} - (R - \epsilon)^{\frac{q-2}{q-1}}\right) \qquad \forall x \in B_R \setminus B_\epsilon,$$

where $C_4(q) = (q-1)^{\frac{q-2}{q-1}}(2-q)^{-1}$, then $-\Delta U_{\epsilon,R} + |\nabla U_{\epsilon,R}|^q \geq 0$. Since u_ϵ vanishes on ∂B_R and is finite on ∂B_ϵ it follows $u_\epsilon \leq U_{\epsilon,R}$ in $B_R \setminus \overline{B}_\epsilon$. Letting successively $\epsilon \to 0$ and $R \to \infty$ yields to (1.3.6). □

Using regularity we can improve this estimate

Lemma 1.3.4 *Under the assumptions of Lemma 1.3.3 there holds*

$$|\nabla u(x)| \leq C_5(q,\Omega)|x|^{-\frac{1}{q-1}} \qquad \forall x \in \Omega. \tag{1.3.7}$$

and

$$u(x) \leq C_6(q,\Omega)d(x)|x|^{-\frac{1}{q-1}} \qquad \forall x \in \Omega. \tag{1.3.8}$$

Proof. For $\ell > 0$, we set

$$T_\ell[u](x) = \ell^{\frac{2-q}{q-1}} u(\ell x) \quad \forall x \in \Omega^\ell := \frac{1}{\ell}\Omega. \tag{1.3.9}$$

If $x \in \Omega$, we set $|x| = d$ and $u_d(y) = T_d[u](y) = d^{\frac{2-q}{q-1}} u(dy)$. Then u_d satisfies (1.1.2) in $\Omega^d = \frac{1}{d}\Omega$. Since $d \leq d^* := \mathrm{diam}(\Omega)$, the curvature of $\partial\Omega^d$ is uniformly bounded and therefore standard a priori estimates imply that there exists c depending on the curvature of Ω^d and $\max\{|u_d(y)| : \frac{1}{2} \leq |y| \leq \frac{3}{2}\}$ such that

$$|\nabla u_d(z)| \leq c \qquad \forall z \in \Omega^d, \frac{3}{4} \leq |z| \leq \frac{5}{4}. \tag{1.3.10}$$

By (1.3.6), c is uniformly bounded. Therefore $|\nabla u(dz)| \leq c d^{-\frac{1}{q-1}}$ which implies (1.3.7).

Next, if $x \in \Omega$ is such that $d(x) \geq \frac{1}{6}|x|$ then (1.3.8) follows from (1.3.6). If $x \in \Omega$ and $d(x) < \frac{1}{6}|x|$, let $P \in \partial\Omega \setminus \{0\}$ such that $|x - P| = d(x)$. By (1.3.7), we obtain

$$u(x) \leq c(q,\Omega)d(x)\int_0^1 |tx + (1-t)P|^{-\frac{1}{q-1}}\, dt, \tag{1.3.11}$$

which, combined with the following estimate

$$|tx + (1-t)P| \geq |x| - (1-t)d(x) \geq \frac{5}{6}|x|$$

implies (1.3.8). □

In the next statement we obtain a local estimate of positive solutions which vanish only on a part of the boundary.

Proposition 1.3.5 *Assume* $1 < q < 2$. *Then there exist* $0 < r^* \leq \delta^*$ *and* $C_7 > 0$ *depending on* N, q *and* Ω *such that for compact set* $K \subset \partial\Omega$, $K \neq \partial\Omega$ *and any positive solution* $u \in C(\overline{\Omega} \setminus K) \cap C^2(\Omega)$ *vanishing on* $\partial\Omega \setminus K$ *of* (1.1.2), *there holds*

$$u(x) \leq C_7 d(x)(d_K(x))^{-\frac{1}{q-1}} \qquad \forall x \in \Omega \quad s.t. \quad d(x) \leq r^*, \tag{1.3.12}$$

where $d_K(x) = \mathrm{dist}\,(x, K)$.

Proof. The proof is based upon the construction of local barriers in spherical shells. We fix $x \in \Omega$ such that $d(x) \leq \delta^*$ and $\sigma(x) := \mathrm{Proj}_{\partial\Omega}(x) \in \partial\Omega \setminus K$. Set $r = d_K(x)$ and consider $\frac{3}{4}r < r' < \frac{7}{8}r$, $\tau \leq 2^{-1}r'$ and $\omega_x = \sigma(x) + \tau\mathbf{n}_x$. Since $\partial\Omega$ is C^2, there exists $r^* \leq \delta^*$, depending only on Ω such that $d_K(\omega_x) > \frac{7}{8}r$ provided $d(x) \leq r^*$. For $A, B > 0$ we define the functions $s \mapsto \tilde{v}(s) = A(r' - s)^{\frac{q-2}{q-1}} - B$ and $y \mapsto v(y) = \tilde{v}(|y - \omega_x|)$ respectively in $[0, r')$ and $B_{r'}(\omega_x)$. Then

$$-\tilde{v}''(s) - \frac{N-1}{s}\tilde{v}'(s) + |\tilde{v}'(s)|^q = A\frac{2-q}{q-1}(r'-s)^{-\frac{q}{q-1}}\left(-\frac{1}{q-1} - \frac{(N-1)(r'-s)}{s} + \left(\frac{(2-q)A}{q-1}\right)^{q-1}\right).$$

We choose A and $\tau > 0$ such that

$$\frac{1}{q-1} - 1 + N + \frac{(N-1)r'}{\tau} \leq \left(\frac{(2-q)A}{q-1}\right)^{q-1} \tag{1.3.13}$$

so that inequality $-\Delta v + |\nabla v|^q \geq 0$ holds in $B_{r'}(\omega_x) \setminus B_\tau(\omega_x)$. We choose B so that $v(\sigma(x)) = \tilde{v}(\tau) = 0$, i.e. $B = A(r' - \tau)^{\frac{q-2}{q-1}}$. Since $\tau \leq \delta^*$, $B_\tau(\omega_x) \subset \Omega^c$ therefore $v \geq 0$ on $\partial\Omega \cap B_{r'}(\omega_x)$ and $v \geq u$ on $\Omega \cap \partial B_{r'}(\omega_x)$. By the maximum principle we obtain that $u \leq v$ in $\Omega \cap B_{r'}(\omega_x)$ and in particular $u(x) \leq v(x)$ i.e.

$$u(x) \leq A\left((r' - \tau - d(x))^{\frac{q-2}{q-1}} - (r'-\tau)^{\frac{q-2}{q-1}}\right) \leq \frac{A(2-q)}{q-1}(r' - \tau - d(x))^{-\frac{1}{q-1}}d(x). \tag{1.3.14}$$

If we take in particular $\tau = \frac{r'}{2}$ and $d(x) \leq \frac{r}{4}$, then $A = A(N, q)$ and

$$u(x) \leq c_9 r'^{-\frac{1}{q-1}} d(x). \tag{1.3.15}$$

where $c_9 = c_9(N, q)$. If we let $r' \to \frac{7}{8}r$ we derive (1.3.12). Next, if $x \in \Omega$ is such that $d(x) \leq \delta^*$ and $d(x) > \frac{1}{4}d_K(x)$, we combine (1.3.12) with Harnack inequality [39], and a standard connectedness argument we obtain that $u(x)$ remains locally bounded in Ω, and the bound on a compact subset G of Ω depends only on K, G, N and q. Since $d_K(x) \geq d(x) > \frac{1}{4}d_K(x)$ it follows from Lemma 1.3.2 that (1.3.12) holds. Finally (1.3.12) holds for every $x \in \Omega$ satisfying $d(x) \leq r^*$. □

As a consequence we have existence of positive solutions of (1.1.2) in Ω with a locally unbounded boundary trace.

Corollary 1.3.6 *Assume* $1 < q < q_c$. *Then for any compact set* $K \subsetneq \partial\Omega$, *there exists a positive solution* u *of* (1.1.2) *in* Ω *such that* $tr_{\partial\Omega}(u) = (\mathcal{S}(u), \mu(u)) = (K, 0)$.

Proof. For any $0 < \epsilon$, we set $K_\epsilon = \{x \in \partial\Omega : d_K(x) < \epsilon\}$ and let ψ_ϵ be a sequence of smooth functions defined on $\partial\Omega$ such that $0 \leq \psi_\epsilon \leq 1$, $\psi_\epsilon = 1$ on K_ϵ, $\psi_\epsilon = 0$ on $\partial\Omega \setminus K_{2\epsilon}$ ($\epsilon < \epsilon_0$ so that $\partial\Omega \setminus K_{2\epsilon} \neq \emptyset$). Furthermore we assume that $\epsilon < \epsilon' < \epsilon_0$ implies $\psi_\epsilon \leq \psi_{\epsilon'}$. For $k \in \mathbb{N}^*$ let $u = u_{k,\epsilon}$ be the solution of

$$\begin{cases} -\Delta u + |\nabla u|^q = 0 & \text{in } \Omega \\ u = k\psi_\epsilon & \text{on } \partial\Omega. \end{cases} \tag{1.3.16}$$

By the maximum principle $(k, \epsilon) \mapsto u_{k,\epsilon}$ is increasing. Combining Proposition 1.3.5 with the same Harnack inequality argument as above we obtain that $u_{k,\epsilon}(x)$ remains locally bounded in Ω and satisfies (1.3.12), independently of k and ϵ. By regularity it remains locally compact in the C^1_{loc}-topology of $\overline{\Omega}\setminus K$. If we set $u_{\infty,\epsilon} := \lim_{k\to\infty} u_{k,\epsilon}$, then it is a solution of (1.1.2) in Ω which satisfies

$$\lim_{x\to y\in K_\epsilon} u_{\infty,\epsilon}(x) = \infty \qquad \forall\, y\in K_\epsilon,$$

locally uniformly in K_ϵ. Furthermore, if $y \in K_\epsilon$ is such that $\overline{B_\theta(y)} \cap \partial\Omega \subset K_\epsilon$ for some $\theta > 0$, then for any k large enough there exists $\theta_k < \theta$ such that

$$\int_{\partial\Omega} \chi_{\overline{B_{\theta_k}(y)}\cap\partial\Omega} dS = k^{-1}.$$

For any $\ell > 0$, $u_{k\ell,\epsilon}$ is minorized by $u := u_{k\ell, B_{\theta_k}(y)\cap\partial\Omega}$ which satisfies

$$\begin{cases} -\Delta u + |\nabla u|^q = 0 & \text{in } \Omega \\ \qquad\qquad u = k\ell\chi_{\overline{B_{\theta_k}(y)}\cap\partial\Omega} & \text{on } \partial\Omega. \end{cases} \tag{1.3.17}$$

When $k\to\infty$, $u_{k\ell,B_{\theta_k}(y)}$ converges to $u_{\ell\delta_y}$ by Theorem 1.2.2 for the stability and Theorem 1.3.17 for the uniqueness. It follows that $u_{\infty,\epsilon} \geq u_{\ell\delta_y}$. Letting $\epsilon\to 0$ and using the same local regularity-compactness argument we obtain that $u_K := u_{\infty,0} = \lim_{\epsilon\to 0} u_{\infty,\epsilon}$ is a positive solution of (1.1.2) in Ω which vanishes on $\partial\Omega\setminus K$ and satisfies

$$u_K \geq u_{\ell\delta_y} \Longrightarrow \lim_{\delta\to 0}\int_{\Sigma_\delta\cap B_\tau(y)} u_K(x)dS \geq \ell,$$

for any $\tau > 0$. Since τ and ℓ are arbitrary, (1.2.37) holds, which implies that $y\in\mathcal{S}(u_K)$. Clearly $\mu(u_K) = 0$ on $\mathcal{R}(u_K) = \partial\Omega\setminus\mathcal{S}(u_K)$ which ends the proof. □

In the supercritical case the above result cannot be always true since there exist removable boundary compact sets (see Section 4). The following result is proved by an easy adaptation of the ideas in the proof of Corollary 1.3.6.

Corollary 1.3.7 *Assume $q_c \leq q < 2$ and let $G\subset\partial\Omega$. We assume that the boundary $\partial_{\partial\Omega}G\subset\partial\Omega$ satisfies the interior boundary sphere condition relative to $\partial\Omega$ in the sense that for any $y\in\partial_{\partial\Omega}G$, there exists $\epsilon_y > 0$ and a sphere such that $B_{\epsilon_y}\cap\partial\Omega\subset G$ and $y\in\overline{B_{\epsilon_y}}$. If $\mathcal{S} := \overline{G} \neq \partial\Omega$ there exists a positive solution u of* (1.1.2) *with boundary trace* $(\mathcal{S}, 0)$.

Remark. It is noticeable that the condition for the singular set to be different from all the boundary is necessary as it is shown in a recent article by Alarcón-García-Melián and Quaas [2]. When $q_c\leq q<2$ and $\Theta\subset\partial\Omega$ it is always possible to construct a positive solution u_ϵ ($\epsilon > 0$) of (1.1.2) with boundary trace $(\Theta^c_\epsilon, 0)$, where $\Theta_\epsilon = \{x\in\partial\Omega : d_\Theta(x) < \epsilon\}$ and the complement is relative to $\partial\Omega$. Furthermore $\epsilon\mapsto u_\epsilon$ is decreasing. If Θ has an empty interior, Proposition 1.3.5 does not apply. We conjecture that $\lim_{\epsilon\to 0} u_\epsilon$ depends on some capacity estimates on Θ.

The condition that a solution vanishes outside a compact boundary set K can be weaken and replaced by a local integral estimate. The next result is fundamental for existence a solution with a given general boundary trace.

Proposition 1.3.8 *Assume* $1 < q < 2$, $U \subset \partial\Omega$ *is relatively open and* $\mu \in \mathfrak{M}(U)$ *is a positive bounded Radon measure. Then for any compact set* $\Theta \subset \Omega$ *there exists a constant* $C_8 = C_8(N, q, H, \Theta, \|\mu\|_{\mathfrak{M}(U)}) > 0$ *such that any positive solution* u *of* (1.1.2) *in* Ω *with boundary trace* $(\mathcal{S}, \mu')$ *where* $\mathcal{S}$ *is closed,* $U \subset \partial\Omega \setminus \mathcal{S} := \mathcal{R}$ *and* μ' *is a positive Radon measure on* $\mathcal{R}$ *such that* $\mu'|_U = \mu$, *there holds*

$$u(x) \leq C_8 \qquad \forall\, x \in \Theta. \tag{1.3.18}$$

Proof. We follow the notations of Theorem 1.2.10. Since the result is local, without loss of generality we can assume that U is smooth and contained in a single chart Σ_j. Estimates (1.2.44)-(1.2.48) are still valid under the form

$$\begin{aligned}\int_U u(\delta,.)\phi^\alpha dS - \int_U u(\delta^*,.)\phi^\alpha dS \\ \geq (1 - c_{10}\epsilon)\int_\delta^{\delta^*}\int_U |\nabla u|^q \phi^\alpha dS(\tau - \delta)d\tau - (\delta^* - \delta)\int_U \tfrac{\partial u}{\partial \delta}(\delta^*,.)\phi^\alpha dS - c'_{10}\end{aligned} \tag{1.3.19}$$

where $c_{10} = c_{10}(q, H)$ and $c'_{10} = c'_{10}(N, q, H)$. Since the second term in the right-hand side of (1.3.19) is uniformly bounded by Lemma 1.3.1, it follows that we can let $\delta \to 0$ and derive,

$$\int_U u(\delta^*,.)\phi^\alpha dS + (1 - c_{10}\epsilon)\int_0^{\delta^*}\int_U |\nabla u|^q \phi^\alpha \tau dS d\tau \leq \int_U \phi^\alpha d\mu + c''_{10} \leq \|\mu\|_{\mathfrak{M}(U)} + c''_{10}, \tag{1.3.20}$$

where c''_{10} depends on the curvature H, N and q. This implies that there exist some ball $B_\alpha(a)$, $\alpha > 0$ and $a \in U$ such that $\overline{B_\alpha(a)} \cap \partial\Omega \subset U$ and

$$\int_{B_\alpha(a)\cap\Omega} |\nabla u|^q d(x) dx \leq \|\mu\|_{\mathfrak{M}(U)} + c''_{10}, \tag{1.3.21}$$

Thus, if $B_\beta(b)$ is some ball such that $\overline{B_\beta(b)} \subset B_\alpha(a) \cap \Omega$, we have

$$\int_{B_\beta(b)} |\nabla u|^q dx \leq (d(b) - \beta)^{-1}\left(\|\mu\|_{\mathfrak{M}(U)} + c''_{10}\right). \tag{1.3.22}$$

If in (1.3.19) we let $\delta \to 0$ and then replace δ^* by $\delta \in (\delta_1, \delta^*]$ for $\delta_1 > 0$ we obtain

$$\int_U \phi^\alpha d\mu \geq \int_U u(\delta,.)\phi^\alpha dS - (\delta^* - \delta)\int_U \frac{\partial u}{\partial \delta}(\delta,.)\phi^\alpha dS - c'''_{10} \tag{1.3.23}$$

where $c'''_{10} = c'''_{10}(N, q, H, \|\mu\|_{\mathfrak{M}(U)})$. By Lemma 1.3.1 the second term in the right-hand side remains bounded by a constant depending on δ_1, H, N and q. Therefore $\int_{U_\Sigma} u(\delta,.)\phi^\alpha dS$ remains bounded by a constant depending on the previous quantities and of $\|\mu\|_{\mathfrak{M}(U)}$ and consequently, assuming that $d(x) \geq \delta_1$ for all $x \in B_\beta(b)$ (i.e. $d(b) - \beta \geq \delta_1$)

$$u_{B_\beta(b)} := \frac{1}{|B_\beta(b)|}\int_{B_\beta(b)} u dx \leq c_{11} \tag{1.3.24}$$

where c_{11} depends on δ_1, H, N, q and $\|\mu\|_{\mathfrak{M}(U)}$. By Poincaré inequality

$$\left(\int_{B_\beta(b)} u^q dx\right)^{\frac{1}{q}} \leq c'_{11}\left(\int_{B_\beta(b)} |\nabla u|^q dx\right)^{\frac{1}{q}} + |B_\beta(b)|^{\frac{1}{q}} u_{B_\beta(b)}. \tag{1.3.25}$$

Combining (1.3.22) and (1.3.25) we derive that $\|u\|_{W^{1,q}(B_\beta(b))}$ remains bounded by a quantity depending only on δ_1, H, N and q and $\|\mu\|_{\mathfrak{M}(U)}$. By the classical trace theorem in Sobolev spaces, $\|u\|_{L^q(\partial B_\beta(b))}$ remains also uniformly bounded when the above quantities are so. By the maximum principle

$$u(x) \leq \mathbb{P}^{B_\beta(b)}[u|_{\partial B_\beta(b)}](x) \qquad \forall\, x \in B_\beta(b), \tag{1.3.26}$$

where $\mathbb{P}^{B_\beta(b)}$ denotes the Poisson kernel in $B_\beta(b)$. Therefore, u remains uniformly bounded in $B_{\frac{\beta}{2}}(b)$ by some constant c''_{11} which also depends on $\|\mu\|_{\mathfrak{M}(U)}$, N, q, Ω, b and β, but not on u. We end the proof by Harnack inequality and a standard connectedness argument as it has already be used in Proposition 1.3.5. □

The main result of this section is the following

Theorem 1.3.9 *Assume* $1 < q < q_c$, $K \subsetneq \partial\Omega$ *is closed and* μ *is a positive Radon measure on* $\mathcal{R} := \partial\Omega \setminus K$. *Then there exists a solution of* (1.1.2) *such that* $tr_{\partial\Omega}(u) = (K, \mu)$.

Proof. For $\epsilon' > \epsilon > 0$ we set $\nu_{\epsilon,\epsilon'} = k\chi_{\overline{K}_{\epsilon'}} + \chi_{\overline{K}^c_\epsilon}\mu$ and denote by $u_{\epsilon,\epsilon',k,\mu}$ the maximal solution of

$$\left\{\begin{aligned} -\Delta u + |\nabla u|^q = 0 \qquad & \text{in } \Omega \\ u = \nu_{\epsilon,\epsilon'} \quad & \text{on } \partial\Omega. \end{aligned}\right. \tag{1.3.27}$$

We recall that $K_\epsilon := \{x \in \partial\Omega : d_K(x) < \epsilon\}$, so that $\nu_{\epsilon,\epsilon'}$ is a positive bounded Radon measure. For $0 < \epsilon \leq \epsilon_0$ there exists $y \in \mathcal{R}$ and $\gamma > 0$ such that $\overline{B}_\gamma(y) \subset \overline{K}^c_{\epsilon_0}$. Since $\left\|\chi_{\overline{K}^c_\epsilon}\mu\right\|_{\mathfrak{M}(\mathcal{R})}$ is uniformly bounded, it follows from Proposition 1.3.8 that $u_{\epsilon,\epsilon',k,\mu}$ remains locally bounded in Ω, uniformly with respect to k, ϵ and ϵ'. Furthermore $(k, \epsilon, \epsilon') \mapsto u_{\epsilon,\epsilon',k,\mu}$ is increasing with respect to k. If $u_{\epsilon,\epsilon',\infty,\mu} = \lim_{k\to\infty} u_{\epsilon,\epsilon',k,\mu}$, it is a solution of (1.1.2) in Ω. By the same argument as the one used in the proof of Corollary 1.3.6, any point $y \in K$ is such that $u_{\epsilon,\epsilon',\infty,\mu} \geq u_{\ell\delta_y}$ for any $\ell > 0$. Using maximum principle

$$(\epsilon_2 \leq \epsilon_1,\, \epsilon'_1 \leq \epsilon'_2,\, k_1 \leq k_2) \Longrightarrow (u_{\epsilon_1,\epsilon'_1,k_1,\mu} \leq u_{\epsilon_2,\epsilon'_2,k_2,\mu}) \tag{1.3.28}$$

Since $u_{\epsilon,\epsilon',\infty,\mu}$ remains locally bounded in Ω independently of ϵ and ϵ', we can set $u_{K,\mu} = \lim_{\epsilon'\to 0}\lim_{\epsilon\to 0} u_{\epsilon,\epsilon',\infty,\mu}$ then by the standard local regularity results $u_{K,\mu}$ is a positive solution of (1.1.2) in Ω. Furthermore $u_{K,\mu} > u_{\ell\delta_y}$, for any $y \in K$ and $\ell > 0$; thus the set of boundary singular points of $u_{K,\mu}$ contains K. In order to prove that $tr_{\partial\Omega}(u_{K,\infty}) = (K, \mu)$ we consider a smooth relatively open set $U \subset \mathcal{R}$. Using the same function ϕ^α as in Proposition 1.3.8, we obtain from (1.3.20)

$$\int_U u_{K,\infty}(\delta^*, .)\phi^\alpha dS + (1 - c_{10}\epsilon)\int_0^{\delta^*}\int_U |\nabla u_{K,\infty}|^q \phi^\alpha \tau dS d\tau \leq \int_U d\mu + c''_{10}. \tag{1.3.29}$$

Therefore U is a subset of the set of boundary regular points of $u_{K,\infty}$, which implies $tr_{\partial\Omega}(u) = (K, \mu)$ by Proposition 1.2.6. □

Remark. If $q_c \leq q < 2$, it is possible to solve (1.3.27) if μ is a smooth function defined in $\mathcal{R}$ and to let successively $k \to \infty$; $\epsilon \to 0$ and $\epsilon' \to 0$ using monotonicity as before. The limit function u^* is a solution of (1.1.2) in Ω. If $tr_{\partial\Omega}(u^*) = (\mathcal{S}^*, \mu^*)$, then $\mathcal{S}^* \subset K$ and $\mu^*|_{\mathcal{R}} = \mu$. However interior points of K, if any, belong to $\mathcal{S}^*$ (see Corollary 1.3.7).

1.3.2 Boundary Harnack inequality

We adapt below ideas from Bauman [5], Bidaut-Véron-Borghol-Véron [7] and Trudinger [38]-[39] in order to prove a *boundary Harnack inequality* which is one of the main tools for analyzing the behavior of positive solutions of (1.1.2) near an isolated boundary singularity. We assume that Ω is a bounded C^2 domain with $0 \in \partial\Omega$ and δ^* has been defined for constructing the flow coordinates.

Theorem 1.3.10 *Assume $0 \in \partial\Omega$, $1 < q < 2$. Then there exist $0 < r_0 \leq \delta^*$ and $C_9 > 0$ depending on N, q and Ω such that for any positive solution $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2r_0})) \cap C^2(\Omega)$ of (1.1.2) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2r_0}$ there holds*

$$\frac{u(y)}{C_9 d(y)} \leq \frac{u(x)}{d(x)} \leq \frac{C_9 u(y)}{d(y)} \tag{1.3.30}$$

for every $x, y \in B_{\frac{2r_0}{3}} \cap \Omega$ satisfying $\frac{|y|}{2} \leq |x| \leq 2|y|$.

Since Ω is a bounded C^2 domain, it satisfies uniform sphere condition, i.e there exists $r_0 > 0$ sufficiently small such that for any $x \in \partial\Omega$ the two balls $B_{r_0}(x - r_0\mathbf{n}_x)$ and $B_{r_0}(x + r_0\mathbf{n}_x)$ are subsets of Ω and $\overline{\Omega}^c$ respectively. We can choose $0 < r_0 < \min\{\delta^*, 3r^*\}$ where r^* is in Proposition 1.3.5.

We first recall the following chained property of the domain Ω [5].

Lemma 1.3.11 *Assume that $Q \in \partial\Omega$, $0 < r < r_0$ and $h > 1$ is an integer. There exists an integer N_0 depending only on r_0 such that for any points x and y in $\Omega \cap B_{\frac{3r}{2}}(Q)$ verifying $\min\{d(x), d(y)\} \geq r/2^h$, there exists a connected chain of balls $B_1, ..., B_j$ with $j \leq N_0 h$ such that*

$$\begin{array}{l} x \in B_1, y \in B_j, \quad B_i \cap B_{i+1} \neq \emptyset \text{ for } 1 \leq i \leq j-1 \\ \text{and } 2B_i \subset B_{2r}(Q) \cap \Omega \text{ for } 1 \leq i \leq j. \end{array} \tag{1.3.31}$$

The next result is an internal Harnack inequality.

Lemma 1.3.12 *Assume $Q \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2r_0}{3}}$ and $0 < r \leq |Q|/4$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2r_0})) \cap C^2(\Omega)$ be a positive solution of (1.1.2) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2r_0}$. Then there exists a positive constant $c_{12} > 1$ depending on N, q, δ^* and r_0 such that*

$$u(x) \leq c_{12}^h u(y), \tag{1.3.32}$$

for every $x, y \in B_{\frac{3r}{2}}(Q) \cap \Omega$ such that $\min\{d(x), d(y)\} \geq r/2^h$ for some $h \in \mathbb{N}$.

Proof. We first notice that for any $\ell > 0$, $T_\ell[u]$ satisfies (1.1.2) in Ω^ℓ where T_ℓ is defined in (1.3.9). If we take in particular $\ell = |Q|$, we can assume $|Q| = 1$ and the curvature of the domain $\Omega^{|Q|}$ remains bounded. By Proposition 1.3.5

$$u(x) \le C_7' \quad \forall x \in B_{2r}(Q) \cap \Omega \tag{1.3.33}$$

where C_7' depends on N, q, δ^*. By Lemma 1.3.11 there exist an integer N_0 depending on r_0 and a connected chain of $j \le N_0 h$ balls B_i with respectively radii r_i and centers x_i, satisfying (1.3.31). Hence due to [38, Corollary 10] and [39, Theorem 1.1] there exists a positive constant c_{12}' depending on N, q, δ^* and r_0 such that for every $1 \le i \le j$,

$$\sup_{B_i} u \le c_{12}' \inf_{B_i} u, \tag{1.3.34}$$

which yields to (1.3.32) with $c_{12} = c_{12}'^{N_0}$. □

Lemma 1.3.13 *Assume the assumptions on Q and u of Lemma 1.3.12 are fulfilled. If $P \in \partial\Omega \cap B_r(Q)$ and $0 < s < r$, there exist two positive constants σ and c_{13} depending on N, q and Ω such that*

$$u(x) \le c_{13} \frac{|x-P|^\sigma}{s^\sigma} M_{s,P}(u) \tag{1.3.35}$$

for every $x \in B_s(P) \cap \Omega$, where $M_{s,P}(u) = \max\{u(z) : z \in B_s(P) \cap \Omega\}$.

Proof. Notice that $B_s(P) \subset B_{2r}(Q)$. Up to the transformation $T_{|Q|}$, we may assume $|Q| = 1$ and hence u is bounded in $B_{2r}(Q) \cap \Omega$ as in (1.3.33). We fix $x \in B_s(P)$ and $s' \in (|x-P|, s)$. Set

$$\tilde{u} := \frac{u}{M_{s,P}(u)}$$

then $M_{s,P}(\tilde{u}) = 1$ and

$$-\Delta\tilde{u} + M_{s,P}^{q-1}(u)\,|\nabla\tilde{u}|^q = 0 \tag{1.3.36}$$

in $B_{2r}(Q) \cap \Omega$. It follows from the assumption and Young's inequality that

$$M_{s,P}^{q-1}(u)\,|\nabla\tilde{u}|^q \le c_{13}'\,|\nabla\tilde{u}|^2 + c_{13}''\,|\nabla\tilde{u}|$$

where c_{13}', c_{13}'' depend on q. By [39, Theorem 5.2] there exist $\sigma = \sigma(N, q, \delta^*) > 0$ and $c_{13} = c_{13}(N, q, \delta^*)$ such that

$$\left|\tilde{u}(z) - \tilde{u}(z')\right| \le c_{13} \left(\frac{s'}{s}\right)^\sigma \quad \forall z, z' \in B_{s'}(P) \cap \Omega,$$

which is equivalent to

$$\left|u(z) - u(z')\right| \le c_{13} \left(\frac{s'}{s}\right)^\sigma M_{s,P}(u) \quad \forall z, z' \in B_{s'}(P) \cap \Omega. \tag{1.3.37}$$

Now by taking $z = x$ and letting $z' \to P$, we obtain

$$u(x) \le c_{13} \left(\frac{s'}{s}\right)^\sigma M_{s,P}(u). \tag{1.3.38}$$

Since (1.3.38) holds true for every $s' > |x-P|$, (1.3.35) follows by rescaling. $\square$

Thanks to Lemma 1.3.12 and Lemma 1.3.13, we obtain the following result by proceeding as in [7, Lemma 5] and [5].

Corollary 1.3.14 *Assume $Q \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2r_0}{3}}$ and $0 < r \leq \frac{|Q|}{8}$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2r_0})) \cap C^2(\Omega)$ be a positive solution of (1.1.2) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2r_0}$. Then there exists a constant c_{14} depending only on N, q, δ^* and r_0 such that*

$$u(x) \leq c_{14} u(Q - \frac{r}{2}\mathbf{n}_Q) \quad \forall x \in B_r(Q) \cap \Omega. \tag{1.3.39}$$

Lemma 1.3.15 *Assume $Q \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2r_0}{3}}$ and $0 < r \leq \frac{|Q|}{8}$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2r_0})) \cap C^2(\Omega)$ be a positive solution of (1.1.2) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2r_0}$. Then there exist $a \in (0, 1/2)$ and $c_{15} > 0$ depending on N, q, δ^* and r_0 such that*

$$\frac{1}{c_{15}} \frac{t}{r} \leq \frac{u(P - t\mathbf{n}_P)}{u(Q - \frac{r}{2}\mathbf{n}_Q)} \leq c_{15} \frac{t}{r} \tag{1.3.40}$$

for any $P \in B_r(Q) \cap \partial\Omega$ and $0 \leq t < \frac{a}{2}r$.

Proof. As above, we may assume $|Q| = 1$, thus estimate (1.3.33) holds.

Step 1 : Lower estimate. Let $0 < \tau < \frac{a}{2}r < \frac{r}{4}$ be fixed. For $b > 0$ to be made precise later on, we define in $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$

$$v(x) = V(s) := \frac{e^{-b\left(\frac{s}{r-\tau}\right)^2} - e^{-\frac{b}{4}}}{e^{-\frac{b}{16}} - e^{-\frac{b}{4}}}, \tag{1.3.41}$$

where $s = \left|x - P + \frac{r}{2}\mathbf{n}_P\right|$. Since $-\Delta v + |\nabla v|^q = -V'' - \frac{N-1}{s}V' + |V'|^q$, this last expression is nonpositive if and only if

$$\begin{aligned} &2b\left(\frac{s}{r-\tau}\right)^2 - N \\ &\quad \geq (2b)^{q-1}\left(e^{-\frac{b}{16}} - e^{-\frac{b}{4}}\right)^{1-q}\left(\frac{s}{r-\tau}\right)^{2(q-1)} s^{2-q} e^{-b(q-1)\left(\frac{s}{r-\tau}\right)^2}. \end{aligned} \tag{1.3.42}$$

Because $x \in B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$,

$$\frac{r-\tau}{4} \leq s \leq \frac{r-\tau}{2} < \frac{1}{4}.$$

Hence, if we choose $b = N + 6$ then (1.3.42) holds true, and thus

$$-\Delta v + |\nabla v|^q \leq 0 \tag{1.3.43}$$

in $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$. Since $B_{r_0}(P - r_0\mathbf{n}_P) \subset \Omega$, it follows that $d(x) \geq dist(x, B_{r_0}(P - r_0\mathbf{n}_P)) \geq r/32$ for any $x \in B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap \partial B_{\frac{r}{4}}(P)$, which, along with Lemma 1.3.12, implies that

$$u(x) \geq c_{15}'^{-1} u(Q - \frac{r}{2}\mathbf{n}_Q) \tag{1.3.44}$$

where
$$c'_{15} = c_{12}^{5}\,(1 + C_7)$$
and C_7 is the constant in (1.3.12). Since $v \le 1$ on $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap \partial B_{\frac{r}{4}}(P)$,
$$u(x) \ge c'^{-1}_{15} u(Q - \frac{r}{2}\mathbf{n}_Q) v(x) \tag{1.3.45}$$
on $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap \partial B_{\frac{r}{4}}(P)$. Moreover $c'^{-1}_{15} u(Q - \frac{r}{2}\mathbf{n}_Q) < c_{12}^{-5} < 1$, therefore $\tilde{v}(x) := c'^{-1}_{15} u(Q - \frac{r}{2}\mathbf{n}_Q) v(x)$ is a subsolution of (1.1.2) in $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$. Consequently, by setting $w := u - \tilde{v}$, we get
$$-\Delta w + d.\nabla w \ge 0$$
in $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$ and $w \ge 0$ on $\partial(B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P))$, where $d = (d_1, ..., d_N)$ and
$$d_i(x) = q\int_0^1 |\nabla(tu + (1-t)\tilde{v})|^{q-2}\,\partial_i(tu + (1-t)\tilde{v})dt \quad \forall 1 \le i \le N.$$
Since $d_i \in L^\infty(B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P))$ for every $i = 1, ..., N$, by applying the maximum principle, we deduce that $u \ge \tilde{v}$ in $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap \partial B_{\frac{r}{4}}(P)$. Finally, set $x_\tau = P - \tau\mathbf{n}_P$ then $x_\tau \in B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$ and
$$v(x_\tau) \ge \frac{e^{-\frac{b}{4}}}{e^{-\frac{b}{16}} - e^{-\frac{b}{4}}}\left(1 - \left(1 - \frac{\tau}{r-\tau}\right)^2\right) \ge c''_{15}\frac{\tau}{r} \tag{1.3.46}$$
where $c''_{15} = c''_{15}(N, q)$, which implies
$$\frac{u(x_\tau)}{u(Q - \frac{r}{2}\mathbf{n}_Q)} \ge c'^{-1}_{15} c''_{15}\frac{\tau}{r}. \tag{1.3.47}$$
Thus the left-hand side of (1.3.40) follows since τ is arbitrary in $(0, \frac{a}{2}r)$.

Step 2 : Upper estimate. Let $a \in (0, 1/2)$ be a parameter to be determined later on. We can choose $r_0 > 0$ so small that $B_{3ar}(P + 3ar\mathbf{n}_P) \subset \overline{\Omega}^c$. Let ϕ_1 be the first eigenfunction of the Laplace operator $-\Delta$ in $B_3 \setminus \overline{B}_1$ with Dirichlet boundary condition and λ_1 is the corresponding eigenvalue. We normalize ϕ_1 by $\phi_1(x) = 1$ on $\{x : |x| = 2\}$ and set
$$\phi_{ar}(x) = \phi_1\left(\frac{x - (P + ar\mathbf{n}_P)}{ar}\right),$$
thus
$$-\Delta\phi_{ar} = \frac{\lambda_1}{(ar)^2}\phi_{ar}(x) \ge 0$$
in $B_{3ar}(P + ar\mathbf{n}_P) \setminus \overline{B}_{ar}(P + ar\mathbf{n}_P)$ and vanishes on the boundary of this domain. We have $-\Delta\phi_{ar} \ge 0 \ge -\Delta u$ in $B_{2ar}(P + ar\mathbf{n}_P) \cap \Omega$. We can choose a small enough such that $B_{2ar}(P + ar\mathbf{n}_P) \subset B_r(Q)$. Then by Corollary 1.3.14,
$$u(x) \le c_{14} u(Q - \frac{r}{2}\mathbf{n}_Q)$$

for $x \in \partial B_{2ar}(P + ar\mathbf{n}_P) \cap \Omega$. Set $\tilde{\phi}_{ar} := c_{14} u(Q - \frac{r}{2}\mathbf{n}_Q)\phi_{ar}$, then $-\Delta\tilde{\phi}_{ar} \geq 0 \geq -\Delta u$ in $B_{2ar}(P + ar\mathbf{n}_P) \cap \Omega$ and $\tilde{\phi}_{ar}$ dominates u on $\partial(B_{2ar}(P + ar\mathbf{n}_P) \cap \Omega)$. By the maximum principle, $u \leq \tilde{\phi}_{ar}$ in $B_{2ar}(P + ar\mathbf{n}_P) \cap \Omega$. In particular

$$u(P - t\mathbf{n}_P) \leq c_{14}\phi_1\left(\frac{|P - t\mathbf{n}_P - (P + ar\mathbf{n}_P)|)}{ar}\right) u(Q - \frac{r}{2}\mathbf{n}_Q).$$

Since $\phi_1(x) \leq c'''_{15} d(x) = c'''_{15}(|x| - 1)$ for every $1 \leq |x| \leq 2$, we obtain the right-hand side of (1.3.40). □

Proof of Theorem 1.3.10. Assume $x \in B_{\frac{2r_0}{3}} \cap \Omega$ and set $r = \frac{|x|}{8}$.

Step 1 : Tangential estimate : $d(x) < \frac{a}{2}r$. Let $Q \in \partial\Omega \setminus \{0\}$ such that $|Q| = |x|$ and $x \in B_r(Q)$. By Lemma 1.3.15,

$$\frac{8}{c_{15}} \frac{u(Q - \frac{r}{2}\mathbf{n}_Q)}{|x|} \leq \frac{u(x)}{d(x)} \leq 8c_{15} \frac{u(Q - \frac{r}{2}\mathbf{n}_Q)}{|x|}. \tag{1.3.48}$$

We can connect $Q - \frac{r}{2}\mathbf{n}_Q$ with $-2r\mathbf{n}_0$ by m_1 (depending only on N) connected balls $B_i = B(x_i, \frac{r}{4})$ with $x_i \in \Omega$ and $d(x_i) \geq \frac{r}{2}$ for every $1 \leq i \leq m_1$. It follows from (1.3.34) that

$$c'^{-m_1}_{12} u(-2r\mathbf{n}_0) \leq u(Q - \frac{r}{2}\mathbf{n}_Q) \leq c'^{m_1}_{12} u(-2r\mathbf{n}_0),$$

which, together with (1.3.48) leads to

$$\frac{8}{c'^{m_1}_{12} c_{15}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq 8c'^{m_1}_{12} c_{15} \frac{u(-2r\mathbf{n}_0)}{|x|}. \tag{1.3.49}$$

Step 2 : Internal estimate : $d(x) \geq \frac{a}{2}r$. We can connect $-2r\mathbf{n}_0$ with x by m_2 (depending only on N) connected balls $B'_i = B(x'_i, \frac{a}{4}r)$ with $x'_i \in \Omega$ and $d(x'_i) \geq \frac{a}{2}r$ for every $1 \leq i \leq m_2$. By applying again (1.3.34) and keeping in mind the estimate $\frac{a}{4}|x| < d(x) \leq |x|$, we get

$$\frac{a}{4c'^{m_2}_{12}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq \frac{4c'^{m_2}_{12}}{a} \frac{u(-2r\mathbf{n}_0)}{|x|}. \tag{1.3.50}$$

Step 3 : End of proof. Take $\frac{|x|}{2} \leq s \leq 2|x|$, we can connect $-2r\mathbf{n}_Q$ with $-s\mathbf{n}_Q$ by m_3 (depending only on N) connected balls $B''_i = B(x''_i, \frac{r}{2})$ with $x''_i \in \Omega$ and $d(x''_i) \geq r$ for every $1 \leq i \leq m_3$. This fact, joint with (1.3.49) and (1.3.50), yields

$$\frac{1}{C'_9} \frac{u(-s\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq C'_9 \frac{u(-s\mathbf{n}_0)}{|x|} \tag{1.3.51}$$

where $C'_9 = C'_9(N, q, \Omega)$. Finally let $y \in B_{\frac{2r_0}{3}} \cap \Omega$ satisfy $\frac{|x|}{2} \leq |y| \leq 2|x|$. By applying twice (1.3.51) we get (1.3.30) with $C_9 = C'^2_9$. □

A direct consequence of Theorem 1.3.10 is the following useful form of boundary Harnack inequality.

Corollary 1.3.16 *Let $u_i \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2r_0})) \cap C^2(\Omega)$ $(i = 1, 2)$ be two nonnegative solutions of* (1.1.2) *vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2r_0}$. Then there exists a constant C_{10} depending on N, q and Ω such that for any $r \leq \frac{2r_0}{3}$*

$$\begin{aligned} \sup\left(\frac{u_1(x)}{u_2(x)} : x \in \Omega \cap (B_r \setminus B_{\frac{r}{2}})\right) \\ \leq C_{10} \inf\left(\frac{u_1(x)}{u_2(x)} : x \in \Omega \cap (B_r \setminus B_{\frac{r}{2}})\right). \end{aligned} \tag{1.3.52}$$

1.3.3 Isolated singularities

Theorem 1.2.2 assert the existence of a solution to (1.2.1) for any positive Radon measure μ if $g \in \mathcal{G}_0$ satisfies (1.2.2), and the question of uniqueness of this problem is still an open question, nevertheless when $\mu = \delta_z$ with $z \in \partial\Omega$, we have the following result

Theorem 1.3.17 *Assume $1 < q < q_c$, $z \in \partial\Omega$ and $c > 0$. Then there exists a unique solution $u := u_{c\delta_z}$ to*

$$\begin{cases} -\Delta u + |\nabla u|^q = 0 & \text{in } \Omega \\ u = c\delta_z & \text{on } \partial\Omega \end{cases} \tag{1.3.53}$$

Furthermore the mapping $c \mapsto u_{c\delta_z}$ is increasing.

Lemma 1.3.18 *Under the assumption of Theorem 1.3.17, there holds*

$$|\nabla u(x)| \leq C_{11} c\, |x - z|^{-N} \quad \forall x \in \Omega \tag{1.3.54}$$

with $C_{11} = C_{11}(N, q, \kappa) > 0$ where κ is the supremum of the curvature of $\partial\Omega$.

Proof. Up to a translation we may assume $z = 0$. By the maximum principle $0 < u(x) \leq cP^{\Omega}(x, 0)$ in Ω. For $0 < \ell \leq 1$, set $v_\ell = T_\ell[u]$ where T_ℓ is the scaling defined in (1.3.9), then v_ℓ satisfies

$$\begin{cases} -\Delta v_\ell + |\nabla v_\ell|^q &= 0 \quad \text{in } \Omega^\ell \\ v_\ell &= \ell^{\frac{2-q}{q-1}+1-N} c\delta_0 \quad \text{on } \partial\Omega^\ell \end{cases} \tag{1.3.55}$$

where $\Omega^\ell = \frac{1}{\ell}\Omega$ and by maximum principle

$$0 < v_\ell(x) \leq \ell^{\frac{2-q}{q-1}+1-N} cP^{\Omega^\ell}(x, 0) \quad \forall x \in \Omega^\ell.$$

Since the curvature of $\partial\Omega^\ell$ remains bounded when $0 < \ell \leq 1$, there holds (see [22])

$$\begin{aligned} &\sup\{|\nabla v_\ell(x)| : x \in \Omega^\ell \cap (B_2 \setminus B_{\frac{1}{2}})\} \\ &\quad \leq C'_{11} \sup\{v_\ell(x) : x \in \Omega^\ell \cap (B_3 \setminus B_{\frac{1}{3}})\} \\ &\quad \leq C'_{11} \ell^{\frac{2-q}{q-1}} \sup\{u(\ell x) : x \in \Omega^\ell \cap (B_3 \setminus B_{\frac{1}{3}})\} \\ &\quad \leq C_{11} c \ell^{\frac{2-q}{q-1}+1-N} \end{aligned} \tag{1.3.56}$$

where C_{11} and C'_{11} depend on N, q and κ. Consequently

$$\ell^{\frac{2-q}{q-1}+1} |\nabla u(\ell x)| \leq C_{11}(N, q, \kappa) c \ell^{\frac{2-q}{q-1}+1-N} \quad \forall x \in \Omega^\ell \cap (B_2 \setminus B_{\frac{1}{2}}), \quad \forall \ell > 0$$

Set $\ell x = y$ and $|x| = 1$, then

$$|\nabla u(y)| \leq C_{11} |y|^{-N} \quad \forall y \in \Omega.$$

□

Lemma 1.3.19

$$\lim_{|x|\to 0} \frac{\mathbb{G}^{\Omega}[|x|^{-Nq}]}{P(x,0)} = 0. \tag{1.3.57}$$

We recall the following estimates for the Green fuction ([7], [16], [40] and [41])

$$G^{\Omega}(x,y) \leq c_{16} d(x) |x-y|^{1-N} \quad \forall x, y \in \Omega, x \neq y$$

and

$$G^{\Omega}(x,y) \leq c_{16} d(x) d(y) |x-y|^{-N} \quad \forall x, y \in \Omega, x \neq y$$

where $c_{16} = c_{16}(N, \Omega)$. Hence, for $\alpha \in (0, N+1-Nq)$, we obtain

$$\begin{aligned} G^{\Omega}(x,y) &\leq \left(c_{16} d(x) |x-y|^{1-N}\right)^{\alpha} \left(c_{16} d(x) d(y) |x-y|^{-N}\right)^{1-\alpha} \\ &= c_{16} d(x) d(y)^{1-\alpha} |x-y|^{\alpha-N} \quad \forall x, y \in \Omega, x \neq y, \end{aligned} \tag{1.3.58}$$

which follows that

$$\frac{\mathbb{G}^{\Omega}[|x|^{-Nq}]}{P^{\Omega}(x,0)} \leq c_{16} |x|^{N} \int_{\mathbb{R}^N} |x-y|^{\alpha-N} |y|^{1-Nq-\alpha} \, dy. \tag{1.3.59}$$

By the following identity (see [23, p. 124]),

$$\int_{\mathbb{R}^N} |x-y|^{\alpha-N} |y|^{1-Nq-\alpha} \, dy = c'_{16} |x|^{1-Nq} \tag{1.3.60}$$

where $c'_{16} = c'_{16}(N, \alpha)$, we obtain

$$\frac{\mathbb{G}^{\Omega}[|x|^{-Nq}]}{P^{\Omega}(x,0)} \leq c_{16} c'_{16} |x|^{N+1-Nq}. \tag{1.3.61}$$

Since $N+1-Nq > 0$, (1.3.57) follows from (1.3.61). □

Proof of Theorem 1.3.17. Since $u = c\,\mathbb{P}^{\Omega}[\delta_0] - \mathbb{G}^{\Omega}[|\nabla u|^q]$,

$$\lim_{|x|\to 0} \frac{u(x)}{P^{\Omega}(x,0)} = c. \tag{1.3.62}$$

Let u and $\tilde{u}$ be two solutions to (1.3.53). For any $\varepsilon > 0$, set $u_{\varepsilon} = (1+\varepsilon)u$ then u_{ε} is a supersolution. By step 3,

$$\lim_{x\to 0} \frac{u_{\varepsilon}(x)}{P^{\Omega}(x,0)} = (1+\varepsilon)c.$$

Therefore there exists $\delta = \delta(\epsilon)$ such that $u_{\epsilon} \geq \tilde{u}$ on $\Omega \cap \partial B_{\delta}$. By the maximum principle, $u_{\varepsilon} \geq \tilde{u}$ in $\Omega \setminus B_{\delta}$. Letting $\varepsilon \to 0$ yields to $u \geq \tilde{u}$ in Ω and the uniqueness follows. The monotonicity of $c \mapsto u_{c\delta_0}$ comes from (1.3.62). □

As a variant of the previous result we have its extension in some unbounded domains.

Theorem 1.3.20 *Assume $1 < q < q_c$ and either $\Omega = \mathbb{R}^N_+ := \{x = (x', x_N) : x_N > 0\}$ or $\partial\Omega$ is compact with $0 \in \partial\Omega$. Then there exists one and only one solution to problem (1.3.53).*

Proof. The proof needs only minor modifications in order to take into account the decay of the solutions at ∞. For $R > 0$ we set $\Omega_R = \Omega \cap B_R$ and denote by $u := u^R_{c\delta_0}$ the unique solution of

$$\left\{\begin{array}{rl} -\Delta u + |\nabla u|^q = 0 & \text{in } \Omega_R \\ u = c\delta_0 & \text{on } \partial\Omega_R. \end{array}\right. \tag{1.3.63}$$

Then

$$u^R_{c\delta_0}(x) \leq cP^{\Omega_R}(x,0) \qquad \forall x \in \Omega_R. \tag{1.3.64}$$

Since $R \mapsto P^{\Omega_R}(.,0)$ is increasing, it follows from (1.3.62) that $R \mapsto u^R_{c\delta_0}$ is increasing too with limit u^* and there holds

$$u^*(x) \leq cP^{\Omega}(x,0) \qquad \forall x \in \Omega. \tag{1.3.65}$$

Estimate (1.3.54) is valid independently of R since the curvature of $\partial\Omega_R$ is bounded (or zero if $\Omega = \mathbb{R}^N_+$). By standard local regularity theory, $\nabla u^R_{c\delta_0}$ converges locally uniformly in $\overline{\Omega} \setminus B_\epsilon$ for any $\epsilon > 0$ when $R \to \infty$, and thus $u^* \in C(\overline{\Omega} \setminus \{0\})$ is a positive solution of (1.1.2) in Ω which vanishes on $\partial\Omega \setminus \{0\}$. It admits therefore a boundary trace $tr_{\partial\Omega}(u^*)$. Estimate (1.3.65) implies that $\mathcal{S}(u^*) = \emptyset$ and $\mu(u^*)$ is a Dirac measure at 0, which is in fact $c\delta_0$ by combining estimates (1.3.62) for Ω_R, (1.3.64) and (1.3.65). Uniqueness follows from the same estimate. □

We next consider the equation (1.1.2) in $\mathbb{R}^N_+$. We denote by $(r,\sigma) \in \mathbb{R}_+ \times S^{N-1}$ are the spherical coordinates in $\mathbb{R}^N$ and we recall the following representation

$$S^{N-1}_+ = \left\{(\sin\phi\sigma', \cos\phi) : \sigma' \in S^{N-2}, \phi \in [0, \frac{\pi}{2})\right\},$$

$$\Delta v = v_{rr} + \frac{N-1}{r}v_r + \frac{1}{r^2}\Delta' v$$

where Δ' is the Laplace-Beltrami operator on S^{N-1},

$$\nabla v = v_r\mathbf{e} + \frac{1}{r}\nabla' v$$

where ∇' denotes the covariant derivative on S^{N-1} identified with the tangential derivative,

$$\Delta' v = \frac{1}{(\sin\phi)^{N-2}}\left((\sin\phi)^{N-2}v_\phi\right)_\phi + \frac{1}{(\sin\phi)^2}\Delta'' v$$

where Δ'' is the Laplace-Beltrami operator on S^{N-2}. Notice that the function $\varphi_1(\sigma) = \cos\phi$ is the first eigenfunction of $-\Delta'$ in $W^{1,2}_0(S^{N-1}_+)$, with corresponding eigenvalue $\lambda_1 = N-1$ and we choose $\theta > 0$ such that $\tilde{\varphi}_1(\sigma) := \theta\cos\phi$ has mass 1 on S^{N-1}_+.

We look for a particular solution of

$$\left\{\begin{array}{rl} -\Delta u + |\nabla u|^q = 0 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ \setminus \{0\} \end{array}\right. \tag{1.3.66}$$

under the separable form

$$u(r,\sigma) = r^{-\beta}\omega(\sigma) \qquad (r,\sigma) \in (0,\infty) \times S_+^{N-1}. \tag{1.3.67}$$

It follows from a straightforward computation that $\beta = \frac{2-q}{q-1}$ and ω satisfies

$$\begin{cases} \mathcal{L}\omega := -\Delta'\omega + \left((\frac{2-q}{q-1})^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} - \frac{2-q}{q-1}(\frac{q}{q-1} - N)\omega & = 0 \quad \text{in } S_+^{N-1} \\ \omega & = 0 \quad \text{on } \partial S_+^{N-1} \end{cases} \tag{1.3.68}$$

Multiplying (1.3.68) by φ_1 and integrating over S_+^{N-1}, we get

$$\left[N - 1 - \frac{2-q}{q-1}\left(\frac{q}{q-1} - N\right)\right] \int_{S_+^{N-1}} \omega\varphi_1 dx + \int_{S_+^{N-1}} \left(\left(\frac{2-q}{q-1}\right)^2 \omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} \varphi_1 dx = 0.$$

Therefore if $N - 1 \geq \frac{2-q}{q-1}\left(\frac{q}{q-1} - N\right)$ *and in particular if* $q \geq q_c$*, there exists no nontrivial solution of* (1.3.68).

In the next theorem we prove that if $N - 1 < \frac{2-q}{q-1}\left(\frac{q}{q-1} - N\right)$, or equivalently $q < \frac{N+1}{N}$, there exists a unique positive solution of (1.3.68).

Theorem 1.3.21 *Assume* $1 < q < q_c$*. There exists a unique positive solution* $\omega_s := \omega \in W^{2,p}(S_+^{N-1})$ *to* (1.3.68) *for all* $p > 1$*. Furthermore* $\omega_s \in C^\infty(\overline{S_+^{N-1}})$*.*

Proof. Step 1 : Existence. We first claim that $\underline{\omega} := \gamma_1\varphi_1^{\gamma_2}$ is a positive sub-solution of (1.3.68) where γ_i $(i = 1,2)$ will be determined later on. Indeed, we have

$$\begin{aligned} \mathcal{L}(\underline{\omega}) &= \gamma_1 \left[(N-1)\gamma_2 - \frac{2-q}{q-1}(\frac{q}{q-1} - N)\right] \varphi_1^{\gamma_2} - \gamma_1\gamma_2(\gamma_2 - 1)\varphi_1^{\gamma_2 - 2} |\nabla'\varphi_1|^2 \\ &\quad + \left[\left(\frac{2-q}{q-1}\right)^2 {\gamma_1}^2{\varphi_1}^{2\gamma_2} + {\gamma_1}^2{\gamma_2}^2{\varphi_1}^{2(\gamma_2-1)} |\nabla'\varphi_1|^2\right]^{\frac{q}{2}} \\ &\leq \gamma_1\varphi_1^{\gamma_2} \left[(N-1)\gamma_2 - \frac{2-q}{q-1}\left(\frac{q}{q-1} - N\right) + 2\left(\frac{2-q}{q-1}\right)^q \gamma_1^{q-1}\varphi_1^{(q-1)\gamma_2}\right] \\ &\quad - \gamma_1\varphi_1^{\gamma_2-2}\left[\left(\frac{2-q}{q-1}\right)^q \gamma_1^{q-1}\varphi_1^{(q-1)\gamma_2+2} + \gamma_2(\gamma_2-1)|\nabla'\varphi_1|^2\right] \\ &\quad + \gamma_1^q\gamma_2^q\varphi_1^{q(\gamma_2-1)}|\nabla'\varphi_1|^q \\ &=: \gamma_1\varphi_1^{\gamma_2} L_1 - \gamma_1\varphi_1^{\gamma_2-2} L_2 + L_3. \end{aligned}$$

Since $q < q_c$, we can choose

$$1 < \gamma_2 < \frac{(N+q-Nq)(2-q)}{(N-1)(q-1)^2}.$$

Since $\varphi_1 \leq 1$, we can choose $\gamma_1 > 0$ small enough in order that $L_1 < 0$. Next, by Young's inequality, for any $\gamma_3 > 0$, we have

$$\varphi_1^{q(\gamma_2-1)-\gamma_2+2} \left|\nabla'\varphi_1\right|^q \leq \gamma_3 \varphi_1^{\frac{2(q(\gamma_2-1)-\gamma_2+2)}{2-q}} + \gamma_3^{\frac{q-2}{q}} \left|\nabla'\varphi_1\right|^2.$$

Since $q > 1$,

$$\frac{2(q(\gamma_2-1)-\gamma_2+2)}{2-q} > (q-1)\gamma_2 + 2,$$

hence

$$\varphi_1^{q(\gamma_2-1)-\gamma_2+2} \left|\nabla'\varphi_1\right|^q \leq \gamma_3 \varphi_1^{(q-1)\gamma_2+2} + \gamma_3^{\frac{q-2}{q}} \left|\nabla'\varphi_1\right|^2.$$

Therefore, if we choose γ_3 such that

$$(\gamma_1\gamma_2)^{\frac{q(q-1)}{2-q}}(\gamma_2-1)^{-\frac{q}{2-q}} < \gamma_3 < \left(\frac{2-q}{q-1}\right)^q \gamma_2^{-q}$$

then $-\gamma_1\varphi_1^{\gamma_2-2}L_2 + L_3 < 0$ and the claim follows.

Next, it is easy to see that $\overline{\omega} = \gamma_4$, with $\gamma_4 > 0$ large enough, is a supersolution of (1.3.68) and $\overline{\omega} > \underline{\omega}$ in $\overline{S}_+^{N-1}$. Therefore there exists a solution $\omega \in W^{2,p}(S_+^{N-1})$ to (1.3.68) such that $0 < \underline{\omega} \leq \omega \leq \overline{\omega}$ in S_+^{N-1}.

Step 2 : Uniqueness. Suppose that ω_1 and ω_2 are two positive different solutions of (1.3.68) and by Hopf lemma $\nabla'\omega_i$ $(i = 1, 2)$ does not vanish on S_+^{N-1}. Up to exchanging the role of ω_1 and ω_2, we may assume $\max_{S_+^{N-1}} \omega_2 \geq \max_{S_+^{N-1}} \omega_1$ and

$$\lambda := \inf\{c > 1 : c\omega_1 > \omega_2 \text{ in } S_+^{N-1}\} > 1.$$

Set $\omega_{1,\lambda} := \lambda\omega_1$, then $\omega_{1,\lambda}$ is a positive supersolution to problem (1.3.68). Owing to the definition of $\omega_{1,\lambda}$, one of two following cases must occur.

Case 1 : Either $\exists\sigma_0 \in S_+^{N-1}$ *such that* $\omega_{1,\lambda}(\sigma_0) = \omega_2(\sigma_0) > 0$ *and* $\nabla'\omega_{1,\lambda}(\sigma_0) = \nabla'\omega_2(\sigma_0)$. Set $\omega_\lambda := \omega_{1,\lambda} - \omega_2$ then $\omega_\lambda \geq 0$ in $\overline{S}_+^{N-1}$, $\omega(\sigma_0) = 0$, $\nabla'\omega_\lambda(\sigma_0) = 0$. Morevover,

$$-\Delta'\omega_\lambda + (H(\omega_{1,\lambda}, \nabla'\omega_{1,\lambda}) - H(\omega_2, \nabla'\omega_2)) - \frac{2-q}{q-1}\left(\frac{q}{q-1} - N\right)\omega_\lambda \geq 0. \qquad (1.3.69)$$

where $H(s,\xi) = ((\frac{2-q}{q-1})^2 s^2 + |\xi|^2)^{\frac{q}{2}}$, $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. By the Mean Value theorem and (1.3.69),

$$-\Delta'\omega_\lambda + \frac{\partial H}{\partial \xi}(\overline{s}, \overline{\xi})\nabla'\omega_\lambda + \left[\frac{\partial H}{\partial s}(\overline{s}, \overline{\xi}) - \frac{2-q}{q-1}\left(\frac{q}{q-1} - N\right)\right]\omega_\lambda \geq 0 \qquad (1.3.70)$$

where $\overline{s}$ and $\overline{\xi}_i$ are the functions with respect to $\sigma \in S_+^{N-1}$. Since $\omega_{1,\lambda}, \omega_2 \in C^1(\overline{S_+^{N-1}})$, we deduce that $\frac{\partial H}{\partial s}(\overline{s}, \overline{\xi}), \left|\frac{\partial H}{\partial \xi}(\overline{s}, \overline{\xi})\right| \in L^\infty(\overline{S_+^{N-1}})$. So we may choose $\gamma_5 > 0$ large enough in other that

$$-\Delta'\omega_\lambda + \frac{\partial H}{\partial \xi}(\overline{s}, \overline{\xi})\nabla'\omega_\lambda + \left[\gamma_5 - \frac{2-q}{q-1}\left(\frac{q}{q-1} - N\right)\right]\omega_\lambda \geq 0.$$

By maximum principle, ω_λ cannot achieve a non-positive minimum in S_+^{N-1}, which is a contradiction.

Case 2 : or $\omega_{1,\lambda} > \omega_2$ *in* S_+^{N-1} *and* $\exists \sigma_0 \in \partial S_+^{N-1}$ *such that*

$$\frac{\partial \omega_{1,\lambda}}{\partial \mathbf{n}}(\sigma_0) = \frac{\partial \omega_2}{\partial \mathbf{n}}(\sigma_0). \tag{1.3.71}$$

Since $\omega_{1,\lambda}(\sigma_0) = 0$ and $\omega_{1,\lambda} \in C^1(\overline{S_+^{N-1}})$, there exists a relatively open subset $U \subset S_+^{N-1}$ such that $\sigma_0 \in \partial U$ and

$$\max_{\overline{U}} w_{1,\lambda} < q^{-\frac{1}{q-1}} \frac{q-1}{2-q} \left(\frac{q}{q-1} - N \right)^{\frac{1}{q-1}}. \tag{1.3.72}$$

We set $\omega_\lambda := \omega_{1,\lambda} - \omega_2$ as in case 1. It follows from (1.3.70) that

$$\begin{aligned} -\Delta' \omega_\lambda + \frac{\partial H}{\partial \xi}(\overline{s}, \overline{\xi}) \partial_{\sigma_i} \omega_\lambda &\geq \left[\frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right) - \frac{\partial H}{\partial s}(\overline{s}, \overline{\omega}) \right] \omega_\lambda \\ &> \frac{2-q}{q-1} \left[\frac{q}{q-1} - N - q \left(\frac{2-q}{q-1} \right)^{q-1} \omega_{1,\lambda}^{q-1} \right] \omega_\lambda > 0 \end{aligned} \tag{1.3.73}$$

in U owing to (1.3.72). By Hopf lemma $\frac{\partial \omega_\lambda}{\partial \mathbf{n}}(\sigma_0) < 0$, which contradicts (1.3.71). The regularity comes from the fact that $\omega^2 + |\nabla \omega|^2 > 0$ in $\overline{S_+^{N-1}}$. □

When $\mathbb{R}_+^N$ is replaced by a general C^2 bounded domain Ω, the role of ω_s is crucial for describing the boundary isolated singularities. In that case we assume that $0 \in \partial\Omega$ and the tangent plane to $\partial\Omega$ at 0 is $\partial\mathbb{R}_+^{N-1} := \{(x', 0) : x' \in \mathbb{R}^{N-1}\}$, with normal inward unit vector $\mathbf{e}_N$. If $u \in C(\overline{\mathbb{R}_+^N} \setminus \{0\})$ is a solution of (1.3.66) then so is $T_\ell[u]$ for any $\ell > 0$. We say that u is *self-similar* if $T_\ell[u] = u$ for every $\ell > 0$.

Proposition 1.3.22 *Assume* $1 < q < q_c$ *and* $0 \in \partial\Omega$. *Then* $u_{\infty,0} := \lim_{c\to\infty} u_{c\delta_0}$ *is a positive solution of* (1.1.2) *in* Ω, *continuous in* $\overline{\Omega} \setminus \{0\}$ *and vanishing on* $\partial\Omega \setminus \{0\}$. *Furthermore there holds*

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} = \sigma \in S_+^{N-1}}} |x|^{\frac{2-q}{q-1}} u_{\infty,0}(x) = \omega_s(\sigma), \tag{1.3.74}$$

locally uniformly on S_+^{N-1}.

Proof. If u is the solution of a problem (1.3.53) in a domain Θ with boundary data $c\delta_z$, we denote it by $u^\Theta_{c\delta_z}$. Let B and B' be two open balls tangent to $\partial\Omega$ at 0 and such that $B \subset \Omega \subset B'^c$. Since $P^B(x,0) \leq P^\Omega(x,0) \leq P^{B'^c}(x,0)$ it follows from Theorem 1.3.20 and (1.3.62) that

$$u^B_{c\delta_0} \leq u^\Omega_{c\delta_0} \leq u^{B'^c}_{c\delta_0}. \tag{1.3.75}$$

Because of uniqueness and whether Θ is B, Ω or B'^c, we have

$$T_\ell[u^\Theta_{c\delta_0}] = u^{\Theta^\ell}_{c\ell^\theta \delta_0} \qquad \forall \ell > 0, \tag{1.3.76}$$

with $\Theta^\ell = \frac{1}{\ell}\Theta$ and $\theta := \frac{2-q}{q-1}+1-N$. Notice also that $c \mapsto u^\Theta_{c\delta_0}$ is increasing. Since $u^\Theta_{c\delta_0}(x) \leq C_4(q)|x|^{\frac{q-2}{q-1}}$ by (1.3.6), it follows that $u^\Theta_{c\delta_0} \uparrow u^\Theta_{\infty,0}$. As in the previous constructions, $u^\Theta_{\infty,0}$ is a positive solution of (1.1.2) in Θ, continuous in $\overline{\Theta}\setminus\{0\}$ and vanishing on $\partial\Theta\setminus\{0\}$.

Step 1 : $\Theta := \mathbb{R}^N_+$. Then $\Theta^\ell = \mathbb{R}^N$. Letting $c \to \infty$ in (1.3.76) yields to

$$T_\ell[u^{\mathbb{R}^N_+}_{\infty,0}] = u^{\mathbb{R}^N_+}_{\infty,0} \qquad \forall \ell > 0. \tag{1.3.77}$$

Therefore $u^{\mathbb{R}^N_+}_{\infty,0}$ is self-similar and thus under the separable form (1.3.67). From Theorem 1.3.21,

$$u^{\mathbb{R}^N_+}_{\infty,0}(x) = |x|^{\frac{q-2}{q-1}}\omega_s(\frac{x}{|x|}). \tag{1.3.78}$$

Step 2 : $\Theta := B$ *or* B'^c. In accordance with our previous notations, we set $B^\ell = \frac{1}{\ell}B$ and $B'^{c\,\ell} = \frac{1}{\ell}B'^c$ for any $\ell > 0$ and we have,

$$T_\ell[u^B_{\infty,0}] = u^{B^\ell}_{\infty,0} \text{ and } T_\ell[u^{B'^c}_{\infty,0}] = u^{B'^{c\,\ell}}_{\infty,0} \tag{1.3.79}$$

and

$$u^{B^{\ell'}}_{\infty,0} \leq u^{B^\ell}_{\infty,0} \leq u^{\mathbb{R}^N_+}_{\infty,0} \leq u^{B'^{c\,\ell}}_{\infty,0} \leq u^{B'^{c\,\ell''}}_{\infty,0} \qquad \forall\, 0 < \ell \leq \ell', \ell'' \leq 1. \tag{1.3.80}$$

When $\ell \to 0$, $u^{B^\ell}_{\infty,0} \uparrow \underline{u}^{\mathbb{R}^N_+}_{\infty,0}$ and $u^{B'^{c\,\ell}}_{\infty,0} \downarrow \overline{u}^{\mathbb{R}^N_+}_{\infty,0}$ where $\underline{u}^{\mathbb{R}^N_+}_{\infty,0}$ and $\overline{u}^{\mathbb{R}^N_+}_{\infty,0}$ are positive solutions of (1.1.2) in $\mathbb{R}^N_+$ such that

$$u^{B^\ell}_{\infty,0} \leq \underline{u}^{\mathbb{R}^N_+}_{\infty,0} \leq u^{\mathbb{R}^N_+}_{\infty,0} \leq \overline{u}^{\mathbb{R}^N_+}_{\infty,0} \leq u^{B'^{c\,\ell}}_{\infty,0} \qquad \forall\, 0 < \ell \leq 1. \tag{1.3.81}$$

This combined with the monotonicity of $u^{B^\ell}_{\infty,0}$ and $u^{B'^{c\,\ell}}_{\infty,0}$ implies that $\underline{u}^{\mathbb{R}^N_+}_{\infty,0}$ and $\overline{u}^{\mathbb{R}^N_+}_{\infty,0}$ vanish on $\partial\mathbb{R}^N_+\setminus\{0\}$ and are continuous in $\overline{\mathbb{R}^N_+}\setminus\{0\}$. Furthermore there also holds for $\ell, \ell' > 0$,

$$T_{\ell'\ell}[u^B_{\infty,0}] = T_{\ell'}[T_\ell[u^B_{\infty,0}]] = u^{B^{\ell\ell'}}_{\infty,0} \text{ and } T_{\ell'\ell}[u^{B'^c}_{\infty,0}] = T_{\ell'}[T_\ell[u^{B'^c}_{\infty,0}]] = u^{B'^{c\,\ell\ell'}}_{\infty,0}. \tag{1.3.82}$$

Letting $\ell \to 0$ and using (1.3.79) and the above convergence, we obtain

$$\underline{u}^{\mathbb{R}^N_+}_{\infty,0} = T_{\ell'}[\underline{u}^{\mathbb{R}^N_+}_{\infty,0}] \text{ and } \overline{u}^{\mathbb{R}^N_+}_{\infty,0} = T_{\ell'}[\overline{u}^{\mathbb{R}^N_+}_{\infty,0}]. \tag{1.3.83}$$

Again this implies that $\underline{u}^{\mathbb{R}^N_+}_{\infty,0}$ and $\overline{u}^{\mathbb{R}^N_+}_{\infty,0}$ are separable solutions of (1.1.2) in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+\setminus\{0\}$ and continuous in $\overline{\mathbb{R}^N_+}\setminus\{0\}$. Therefore they coincide with $u^{\mathbb{R}^N_+}_{\infty,0}$.

Step 3 : End of the proof. From (1.3.75) and (1.3.79) there holds

$$u^{B^\ell}_{\infty,0} \leq T_\ell[u^\Omega_{\infty,0}] \leq u^{B'^{c\,\ell}}_{\infty,0} \qquad \forall\, 0 < \ell \leq 1. \tag{1.3.84}$$

Since the left-hand side and the right-hand side of (1.3.84) converge to the same function $u^{\mathbb{R}^N_+}_{\infty,0}(x)$, we obtain

$$\lim_{\ell\to 0} \ell^{\frac{2-q}{q-1}} u^\Omega_{\infty,0}(\ell x) = |x|^{\frac{q-2}{q-1}}\omega_s(\frac{x}{|x|}) \tag{1.3.85}$$

and this convergence holds in any compact subset of Ω. If we fix $|x| = 1$, we derive (1.3.74). □

Remark. It is possible to improve the convergence in (1.3.74) by straightening $\partial\Omega$ near 0 (and thus to replace $u^{\Omega}_{\infty,0}$ by a function $\tilde{u}^{\Omega}_{\infty,0}$ defined in $B_\epsilon \cap \mathbb{R}^N_+$) and to obtain a convergence in $C^1(\overline{S^{N-1}_+})$.

Combining this result with Theorem 1.2.11 we derive

Corollary 1.3.23 *Assume* $1 < q < q_c$ *and* $0 \in \partial\Omega$. *If* u *is a positive solution of* (1.1.2) *with boundary trace* $tr_{\partial\Omega}(u) = (\mathcal{S}(u), \mu(u)) = (\{0\}, 0)$ *then* $u \geq u^{\Omega}_{\infty,0}$.

The next result asserts the existence of a maximal solution with boundary trace $(\{0\}, 0)$.

Proposition 1.3.24 *Assume* $1 < q < q_c$ *and* $0 \in \partial\Omega$. *Then there exists a maximal solution* $U := U^{\Omega}_{\infty,0}$ *of* (1.1.2) *with boundary trace* $tr_{\partial\Omega}(U) = (\mathcal{S}(U), \mu(U)) = (\{0\}, 0)$. *Furthermore*

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} = \sigma \in S^{N-1}_+}} |x|^{\frac{2-q}{q-1}} U^{\Omega}_{\infty,0}(x) = \omega_s(\sigma), \tag{1.3.86}$$

locally uniformly on S^{N-1}_+.

Proof. Step 1 : Existence. Since $1 < q < q_c < \frac{N}{N-1}$, there exists a radial separable singular solution of (1.1.2) in $\mathbb{R}^N \setminus \{0\}$,

$$U_S(x) = \Lambda_{N,q} |x|^{\frac{q-2}{q-1}} \quad \text{with} \quad \Lambda_{N,q} = \left(\frac{q-1}{2-q}\right)^{q'} \left(\frac{(2-q)(N-(N-1)q)}{(q-1)^2}\right)^{\frac{1}{q-1}}. \tag{1.3.87}$$

By Lemma 1.3.3 there exists $C_4(q) > 0$ such that any positive solution u of (1.1.2) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ satisfies $u(x) \leq C_4(q)|x|^{\frac{q-2}{q-1}}$ in Ω. Therefore, $U^*(x) = \Lambda^*|x|^{\frac{q-2}{q-1}}$ with $\Lambda^* := \Lambda^*(N, q) \geq \max\{\Lambda_{N,q}, C_4(q)\}$ is a supersolution of (1.1.2) in $\mathbb{R}^N \setminus \{0\}$ and dominates in Ω any solution u vanishing on $\partial\Omega \setminus \{0\}$. For $0 < \epsilon < \max\{|z| : z \in \Omega\}$, we denote by u_ϵ the solution of

$$\begin{cases} -\Delta u_\epsilon + |\nabla u_\epsilon|^q = 0 & \text{in } \Omega \setminus B_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega \setminus B_\epsilon \\ u_\epsilon = \Lambda^* \epsilon^{\frac{q-2}{q-1}} & \text{on } \Omega \cap \partial B_\epsilon. \end{cases} \tag{1.3.88}$$

If $\epsilon' < \epsilon$, $u_{\epsilon'}|_{\partial(\Omega \setminus B_\epsilon)} \leq u_\epsilon|_{\partial(\Omega \setminus B_\epsilon)}$, therefore

$$u \leq u_{\epsilon'} \leq u_\epsilon \leq U^*(x) \qquad \text{in } \Omega. \tag{1.3.89}$$

Letting ϵ to zero, $\{u_\epsilon\}$ decreases and converges to some $U^{\Omega}_{\infty,0}$ which vanishes on $\partial\Omega \setminus \{0\}$. By the the regularity estimates already used in stability results, the convergence occurs in $C^1_{loc}(\overline{\Omega} \setminus \{0\})$, $U^{\Omega}_{\infty,0} \in C(\overline{\Omega} \setminus \{0\})$ is a positive solution of (1.1.2) and it belongs to $C^2(\Omega)$; furthermore it has boundary trace $(\{0\}, 0)$ and for any positive solution u satisfying $tr_{\partial\Omega}(u) = (\{0\}, 0)$ there holds

$$u^{\Omega}_{\infty,0} \leq u \leq U^{\Omega}_{\infty,0} \leq U^*(x). \tag{1.3.90}$$

Therefore $U^{\Omega}_{\infty,0}$ is the maximal solution.

Step 2 : $\Omega=\mathbb{R}^N_+$. Since

$$T_\ell[U^*]\big|_{|x|=\epsilon}=U^*\big|_{|x|=\epsilon}\qquad\forall\,\ell>0, \tag{1.3.91}$$

there holds

$$T_\ell[u_\epsilon]=u_{\frac{\epsilon}{\ell}}. \tag{1.3.92}$$

Letting $\epsilon\to 0$ yields to $T_\ell[U^{\mathbb{R}^N_+}_{\infty,0}]=U^{\mathbb{R}^N_+}_{\infty,0}$. Therefore $U^{\mathbb{R}^N_+}_{\infty,0}$ is self-similar and coincide with $u^{\mathbb{R}^N_+}_{\infty,0}$.

Step 3 : $\Omega=B$ or B'^c. We first notice that the maximal solution is an increasing function of the domain. Since $T_\ell[u^\Theta_\epsilon]=u^{\Theta^\ell}_{\frac{\epsilon}{\ell}}$ where we denote by u^Θ_ϵ the solution of (1.3.88) in $\Theta\setminus B_\epsilon$ for any $\ell,\epsilon>0$ and any domain Θ (with $0\in\partial\Theta$), we derive as in Proposition 1.3.22-Step 2, using (1.3.92) and uniqueness,

$$T_\ell[U^B_{\infty,0}]=U^{B^\ell}_{\infty,0}\ \text{and}\ \ T_\ell[U^{B'^c}_{\infty,0}]=U^{B'^c\,\ell}_{\infty,0} \tag{1.3.93}$$

and

$$U^{B^{\ell'}}_{\infty,0}\leq U^{B^\ell}_{\infty,0}\leq u^{\mathbb{R}^N_+}_{\infty,0}\leq U^{B'^c\,\ell}_{\infty,0}\leq U^{B'^c\,\ell''}_{\infty,0}\qquad\forall\,0<\ell\leq\ell',\ell''\leq 1. \tag{1.3.94}$$

As in Proposition 1.3.22, $U^{B^\ell}_{\infty,0}\uparrow\underline{U}^{\mathbb{R}^N_+}_{\infty,0}\leq U^{\mathbb{R}^N_+}_{\infty,0}$ and $U^{B'^c\,\ell}_{\infty,0}\downarrow\overline{U}^{\mathbb{R}^N_+}_{\infty,0}\geq U^{\mathbb{R}^N_+}_{\infty,0}$ where $\underline{U}^{\mathbb{R}^N_+}_{\infty,0}$ and $\overline{U}^{\mathbb{R}^N_+}_{\infty,0}$ are positive solutions of (1.1.2) in $\mathbb{R}^N$ which vanish on $\partial\mathbb{R}^N_+\setminus\{0\}$ and endow the same scaling invariance under T_ℓ. Therefore they coincide with $u^{\mathbb{R}^N_+}_{\infty,0}$.

Step 3 : End of the proof. It is similar to the one of Proposition 1.3.22. □

Combining Proposition 1.3.22 and Proposition 1.3.24 we can prove the final result :

Theorem 1.3.25 *Assume $1<q<q_c$ and $0\in\partial\Omega$. Then $U^\Omega_{\infty,0}=u^\Omega_{\infty,0}$.*

Proof. We follow the method used in [16, Sec 4].

Step 1 : Straightening the boundary. We represent $\partial\Omega$ near 0 as the graph of a C^2 function ϕ defined in $\mathbb{R}^{N-1}\cap B_R$ and such that $\phi(0)=0$, $\nabla\phi(0)=0$ and

$$\partial\Omega\cap B_R=\{x=(x',x_N):x'\in\mathbb{R}^{N-1}\cap B_R,x_N=\phi(x')\}.$$

We introduce the new variable $y=\Phi(x)$ with $y'=x'$ and $y_N=x_N-\phi(x')$, with corresponding spherical coordinates in $\mathbb{R}^N$, $(r,\sigma)=(|y|,\frac{y}{|y|})$. If u is a positive solution of (1.1.2) in Ω vanishing on $\partial\Omega\setminus\{0\}$, we set $\tilde u(y)=u(x)$, then a technical computation shows that $\tilde u$ satisfies with $\mathbf{n}=\frac{y}{|y|}$

$$\begin{aligned}
&r^2\tilde u_{rr}\Big(1-2\phi_r\langle\mathbf{n},\mathbf{e}_N\rangle+|\nabla\phi|^2\langle\mathbf{n},\mathbf{e}_N\rangle^2\Big)\\
&+r\tilde u_r\Big(N-1-r\langle\mathbf{n},\mathbf{e}_N\rangle\Delta\phi-2\langle\nabla'\langle\mathbf{n},\mathbf{e}_N\rangle,\nabla'\phi\rangle+r|\nabla\phi|^2\langle\nabla'\langle\mathbf{n},\mathbf{e}_N\rangle,\mathbf{e}_N\rangle\Big)\\
&+\langle\nabla'\tilde u,\mathbf{e}_N\rangle\Big(2\phi_r-|\nabla\phi|^2\langle\mathbf{n},\mathbf{e}_N\rangle-r\Delta\phi\Big)\\
&+r\langle\nabla'\tilde u_r,\mathbf{e}_N\rangle\Big(2\langle\mathbf{n},\mathbf{e}_N\rangle|\nabla\phi|^2-2\phi_r\Big)-2\langle\nabla'\tilde u_r,\nabla'\phi\rangle\langle\mathbf{n},\mathbf{e}_N\rangle\\
&+|\nabla\phi|^2\langle\nabla'\langle\nabla'\tilde u,\mathbf{e}_N\rangle,\mathbf{e}_N\rangle-\tfrac{2}{r}\langle\nabla'\langle\nabla'\tilde u,\mathbf{e}_N\rangle,\nabla'\phi\rangle+\Delta'\tilde u\\
&+r^2\left|\tilde u_r\mathbf{n}+\tfrac{1}{r}\nabla'\tilde u-(\phi_r\mathbf{n}+\tfrac{1}{r}\nabla'\phi)\langle\tilde u_r\mathbf{n}+\tfrac{1}{r}\nabla'\tilde u,\mathbf{e}_N\rangle\right|^q=0.
\end{aligned}\tag{1.3.95}$$

Using the transformation $t = \ln r$ for $t \leq 0$ and $\tilde{u}(r,\sigma) = r^{\frac{q-2}{q-1}} v(t,\sigma)$, we obtain finally that v satisfies

$$
\begin{aligned}
&(1+\epsilon_1)\, v_{tt} + \left(N - \tfrac{2}{q-1} + \epsilon_2\right) v_t + (\lambda_{N,q} + \epsilon_3)\, v + \Delta' v \\
&\quad + \langle \nabla' v, \overrightarrow{\epsilon_4} \rangle + \langle \nabla' v_t, \overrightarrow{\epsilon_5} \rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \overrightarrow{\epsilon_6} \rangle \\
&\quad - \left| (\tfrac{q-2}{q-1} v + v_t)\mathbf{n} + \nabla' \tilde{v} + \langle (\tfrac{q-2}{q-1} v + v_t)\mathbf{n} + \nabla' \tilde{v}, \mathbf{e}_N \rangle \overrightarrow{\epsilon}_7 \right|^q = 0,
\end{aligned}
\tag{1.3.96}
$$

on $(-\infty, \ln R] \times S_+^{N-1} := Q_R$ and vanishes on $(-\infty, \ln R] \times \partial S_+^{N-1}$, where

$$
\lambda_{N,q} = \left(\frac{2-q}{q-1} \right) \left(\frac{q}{q-1} - N \right).
$$

Furthermore the ϵ_j are uniformly continuous functions of t and $\sigma \in S^{N-1}$ for $j = 1, ..., 7$, C^1 for $j = 1, 5, 6, 7$ and satisfy the following decay estimates

$$
|\epsilon_j(t,.)| \leq Ce^t \quad \text{for } j = 1, ..., 7 \text{ and } |\epsilon_{j\,t}(t,.)| + |\nabla' \epsilon_j| \leq c_{17} e^t \quad \text{for } j = 1, 5, 6, 7.
\tag{1.3.97}
$$

Since v, v_t and $\nabla' v$ are uniformly bounded and by standard regularity methods of elliptic equations [16, Lemma 4.4], there exist a constant $c'_{17} > 0$ and $T < \ln R$ such that

$$
\|v(t,.)\|_{C^{2,\gamma}(\overline{S_+^{N-1}})} + \|v_t(t,.)\|_{C^{1,\gamma}(\overline{S_+^{N-1}})} + \|v_{tt}(t,.)\|_{C^{0,\gamma}(\overline{S_+^{N-1}})} \leq c'_{17}
\tag{1.3.98}
$$

for any $\gamma \in (0,1)$ and $t \leq T - 1$. Consequently the set of functions $\{v(t,.)\}_{t \leq 0}$ is relatively compact in the $C^2(\overline{S_+^{N-1}})$ topology and there exist η and a subsequence $\{t_n\}$ tending to $-\infty$ such that $v(t_n,.) \to \eta$ when $n \to \infty$ in $C^2(\overline{S_+^{N-1}})$.

Step 2 : End of the proof. Taking $u = u_{\infty,0}^{\Omega}$ or $u = U_{\infty,0}^{\Omega}$, with corresponding v, we already know that $v(t,.)$ converges to ω_s, locally uniformly on S_+^{N-1}. Thus ω_s is the unique element in the limit set of $\{v(t,.)\}_{t \leq 0}$ and $\lim_{t \to -\infty} v(t,.) = \omega_s$ in $C^2(\overline{S_+^{N-1}})$. This implies in particular

$$
\lim_{x \to 0} \frac{u_{\infty,0}^{\Omega}(x)}{U_{\infty,0}^{\Omega}(x)} = 1
\tag{1.3.99}
$$

and uniqueness follows from maximum principle. □

As a consequence we have a full characterization of positive solution with an isolated boundary singularity

Corollary 1.3.26 *Assume* $1 < q < q_c$, $0 \in \partial\Omega$ *and* $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ *is a nonnegative solution of* (1.1.2) *vanishing on* $\partial\Omega \setminus \{0\}$. *Then either there exists* $c \geq 0$ *such that* $u = u_{c\delta_0}$, *or* $u = u_{\infty,0}^{\Omega} = \lim_{c \to \infty} u_{c\delta_0}$.

1.4 The supercritical case

In this section we consider the case $q_c \leq q < 2$.

1.4.1 Removable isolated singularities

Theorem 1.4.1 *Assume $q_c \leq q < 2$, $0 \in \partial\Omega$ and $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a nonnegative solution of* (1.1.2) *vanishing on $\partial\Omega \setminus \{0\}$. Then $u \equiv 0$.*

Proof. Step 1 : Integral estimates. We consider a sequence of functions $\zeta_n \in C^\infty(\mathbb{R}^N)$ such that $\zeta_n(x) = 0$ if $|x| \leq \frac{1}{n}$, $\zeta_n(x) = 1$ if $|x| \geq \frac{2}{n}$, $0 \leq \zeta_n \leq 1$ and $|\nabla\zeta_n| \leq c_{18}n$, $|\Delta\zeta_n| \leq c_{18}n^2$ where c_{18} is independent of n. As a test function we take $\xi\zeta_n$ (where ξ is the solution to (1.2.14)) and we obtain

$$\int_\Omega (|\nabla u|^q \xi\zeta_n - u\zeta_n\Delta\xi)\, dx = \int_\Omega u\, (\xi\Delta\zeta_n + 2\nabla\xi.\nabla\zeta_n)\, dx = I + II. \tag{1.4.1}$$

Set $\Omega_n = \Omega \cap \{x : \frac{1}{n} < |x| \leq \frac{2}{n}\}$, then $|\Omega_n| \leq c'_{18}(N)n^{-N}$, thus

$$I \leq c_{18}C_4(q)\int_{\Omega_n} n^{\frac{2-q}{q-1}+2}\xi dx \leq c''_{18}n^{\frac{2-q}{q-1}+2-1-N} = c''_{18}n^{\frac{1}{q-1}-\frac{1}{q_c-1}}$$

since $\xi(x) \leq c_3 d(x)$. Moreover,

$$II \leq c_{18}C_4(q)\int_{\Omega_n} n^{\frac{2-q}{q-1}+1}|\nabla\xi| dx \leq c_{19}n^{\frac{2-q}{q-1}+1-N} = c_{19}n^{\frac{1}{q-1}-\frac{1}{q_c-1}}.$$

Since $\frac{1}{q-1} - \frac{1}{q_c-1} \leq 0$, the right-hand side of (1.4.1) remains uniformly bounded, hence it follows from monotone convergence theorem that

$$\int_\Omega (|\nabla u|^q\xi + u)\, dx < \infty. \tag{1.4.2}$$

More precisely, if $q > q_c$, $I + II$ goes to 0 as $n \to \infty$ which implies

$$\int_\Omega (|\nabla u|^q\xi + u)\, dx = 0.$$

Next we assume $q = q_c$. Since $|\nabla u| \in L^{q_c}_d(\Omega)$, $v := \mathbb{G}^\Omega[|\nabla u|^{q_c}] \in L^1(\Omega)$. Furthermore, $u + v$ is positive and harmonic in Ω. Its boundary trace is a Radon measure and since the boundary trace $Tr(v)$ of v is zero, there exists $c \geq 0$ such that $Tr(u) = c\delta_0$. Equivalently, u solves the problem

$$\begin{cases} -\Delta u + |\nabla u|^{q_c} = 0 & \text{in } \Omega \\ u = c\delta_0 & \text{on } \partial\Omega. \end{cases} \tag{1.4.3}$$

Furthermore, since $u \in L^1(\Omega)$, $u(x) \leq cP(x,.)$ in Ω. Therefore, if $c = 0$, so is u. Let us assume that $c > 0$.

Step 2 : The flat case. Assume $\Omega = B_1^+ := B_1 \cap \mathbb{R}^N_+$. We use the spherical coordinates $(r,\sigma) \in [0,\infty) \times S^{N-1}$ as above. Put

$$\overline{f} = \int_{S^{N-1}_+} f\tilde{\varphi}_1 dS$$

then

$$\overline{u}_{rr} + \frac{N-1}{r}\overline{u}_r - \frac{N-1}{r^2}\overline{u} = \overline{|\nabla u|^{q_c}} \tag{1.4.4}$$

Set $v(r) = r^{N-1}\overline{u}(r)$, then

$$v_{rr} + \frac{1-N}{r}v_r = r^{N-1}\overline{|\nabla u|^{q_c}}. \tag{1.4.5}$$

and

$$v_r(r) = r^{N-1}v_r(1) - r^{N-1}\int_r^1 \overline{|\nabla u|^{q_c}}(s)ds. \tag{1.4.6}$$

Since

$$\int_0^1 r^{N-1}\int_r^1 \overline{|\nabla u|^{q_c}}(s)ds = \frac{1}{N}\int_0^1 r^N\overline{|\nabla u|^{q_c}}(s)ds < \infty \tag{1.4.7}$$

it follows that there exists $\lim_{r\to 0} v(r) = \alpha \geq 0$. Let us assume that $\alpha > 0$. From (1.4.5),

$$(r^{1-N}v_r)_r = \overline{|\nabla u|^{q_c}} > 0$$

then

$$r_1^{1-N}v_r(r_1) = r_2^{1-N}v_r(r_2) + \int_{r_2}^{r_1}\overline{|\nabla u|^{q_c}}ds \quad \forall 0 < r_2 < r_1. \tag{1.4.8}$$

This implies that $v_r(r)$ keeps a constant sign on $(0, r_1)$ for some $r_1 > 0$. If $v_r < 0$, then

$$\overline{u}_r = \left((1-N)v + rv_r\right)r^{-N} \Longrightarrow \overline{|\nabla u|^{q_c}} \geq \left(\frac{(N-1)\alpha}{2}r^{-N}\right)^{q_c} \quad \forall 0 < r < r_2, \tag{1.4.9}$$

for some $0 < r_2 < r_1$. It follows that $\overline{|\nabla u|^{q_c}} \notin L^1_d(B_1^+)$, which is a contradiction. Thus $v_r > 0$. By (1.4.6)

$$\int_0^1 \overline{|\nabla u|^{q_c}}(s)ds \leq v_r(1).$$

Using again (1.4.6) it implies $\lim_{r\to 0} v_r(r) = 0$. Thus (1.4.9) applies and we get the same contradiction. Therefore $\alpha = 0$, equivalently

$$\lim_{r\to 0} r^{N-1}\int_{S_+^{N-1}} u(r,\sigma)\tilde{\varphi}_1(\sigma)dS = 0. \tag{1.4.10}$$

Set $\Gamma := \{\sigma = (\sigma', \phi) \in S_+^{N-1} : 0 \leq \phi \leq \frac{\pi}{4}\}$, then

$$\lim_{r\to 0} r^{N-1}\int_\Gamma u(r,\sigma)dS = 0. \tag{1.4.11}$$

By Harnack inequality in Theorem 1.3.10 and since $\tilde{\varphi} \leq \gamma$

$$\gamma^{-1}u(r,\tau) \leq \frac{u(r,\tau)}{\tilde{\varphi}_1(\tau)} \leq c_{20}u(r,\sigma) \quad \forall(\tau,\sigma) \in S_+^{N-1}\times\Gamma. \tag{1.4.12}$$

Integrating over Γ and using (1.4.11) it follows

$$\lim_{x\to 0} |x|^N\frac{u(x)}{d(x)} = 0. \tag{1.4.13}$$

By standard regularity methods, (1.4.12) can be improved in order to take into account that u vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and we get

$$\lim_{x\to 0} |x|^N \frac{u(x)}{d(x)} = 0 \Longleftrightarrow \lim_{x\to 0} \frac{u(x)}{P^{\mathbb{R}^N_+}(x,0)} = 0, \tag{1.4.14}$$

where $P^{\mathbb{R}^N_+}(x,0)$ is the Poisson kernel in $\mathbb{R}^N_+$ with singularity at 0. Since $P^{\mathbb{R}^N_+}(.,0)$ is a super solution and $u = o(P^{\mathbb{R}^N_+}(.,0))$, the maximum principle implies $u = 0$.

Step 3 : The general case. For $\ell > 0$, we set

$$v_\ell(x) = T_\ell[u](x) = \ell^{N-1}u(\ell x).$$

Then v_ℓ satisfies

$$\left\{\begin{array}{ll} -\Delta v_\ell + |\nabla v_\ell|^{q_c} = 0 & \text{in } \Omega^\ell \\ v_\ell = 0 & \text{on } \partial\Omega^\ell \setminus \{0\}. \end{array}\right. \tag{1.4.15}$$

Furthermore, $T_\ell[P^\Omega] = P^{\Omega^\ell}$ with $P^\Omega := P^{\Omega^1}$ and

$$u(x) \le cP^\Omega(x,0) \quad \forall x \in \Omega \Longrightarrow v_\ell(x) \le cP^{\Omega^\ell}(x,0) \quad \forall x \in \Omega^\ell.$$

By standard a priori estimates [22], for any $R > 0$ there exists $M(N,q,R) > 0$ such that, if $\Gamma_R = B_{2R} \setminus B_R$,

$$\sup\left\{|v_\ell(x)| + |\nabla v_\ell(x)| : x \in \Gamma_R \cap \Omega^\ell\right\} + \sup\left\{\frac{|\nabla v_\ell(x) - \nabla v_\ell(y)|}{|x-y|^\gamma} : (x,y) \in \Gamma_R \cap \Omega^\ell\right\} \le M(N,q,R), \tag{1.4.16}$$

where $\gamma \in (0,1)$ is independent of $\ell \in (0,1]$. Notice that these uniform estimates, up to the boundary, holds because the curvature of $\partial\Omega^\ell$ remains uniformly bounded when $\ell \in (0,1]$. By compactness, there exist a sequence $\{\ell_n\}$ converging to 0 and function $v \in C^1(\overline{\mathbb{R}^N_+}\setminus\{0\})$ such that

$$\sup\left\{|(v_{\ell_n} - v)(x)| + |\nabla(v_{\ell_n} - v)(x)| : x \in \Gamma_R \cap \Omega^{\ell_n}\right\} \to 0$$

Furthermore v satisfies

$$\left\{\begin{array}{ll} -\Delta v + |\nabla v|^{q_c} = 0 & \text{in } \mathbb{R}^N_+ \\ v = 0 & \text{on } \partial\mathbb{R}^N_+ \setminus \{0\}. \end{array}\right. \tag{1.4.17}$$

From step 2, $v = 0$ and

$$\sup\left\{|v_{\ell_n}(x)| + |\nabla v_{\ell_n}(x)| : x \in \Gamma_R \cap \Omega^{\ell_n}\right\} \to 0;$$

therefore

$$\lim_{x\to 0} |x|^{N-1} u(x) = 0 \qquad \text{and} \qquad \lim_{x\to 0} |x|^N |\nabla u(x)| = 0. \tag{1.4.18}$$

Integrating from $\partial\Omega$, we obtain

$$\lim_{x\to 0} \frac{|x|^N}{d(x)} u(x) = 0. \tag{1.4.19}$$

Equivalently $u(x) = o(P^\Omega(x,0))$ which implies $u = 0$ by the maximum principle. □

1.4.2 Removable singularities

The next statement, valid for a positive solution of

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1.4.20}$$

is easy to prove :

Proposition 1.4.2 *Let u be a positive solution of* (1.1.2)*. The following assertions are equivalent :*

(i) u is moderate (see Definition 1.1.8).

(ii) $u \in L^1(\Omega)$, $|\nabla u| \in L^q_d(\Omega)$.

(iii) The boundary trace of u is a positive bounded measure μ on $\partial\Omega$.

Let φ be the first eigenfunction of $-\Delta$ in $W^{1,2}_0(\Omega)$ normalized so that $\sup_\Omega \varphi = 1$ and λ be the corresponding eigenvalue. We start with the following simple result.

Lemma 1.4.3 *Let Ω be a bounded C^2 domain. Then for any $q \geq 1$, $0 \leq \alpha < 1$, $\gamma \in [0, \delta^*)$ and $u \in C^1(\Omega)$, there holds*

$$\begin{aligned} &\int_{\gamma<d(x)<\delta^*} (d(x)-\gamma)^{-\alpha}|u|^q dx \\ &\qquad \leq C_{12}\left(\int_\Sigma |u(\delta^*,\sigma)|^q dS + \int_{\gamma<d(x)<\delta^*} (d(x)-\gamma)^{q-\alpha}|\nabla u|^q dx\right) \end{aligned} \tag{1.4.21}$$

where $C_{12} = C_{12}(\alpha, q, \Omega)$. If $1 < q < 2$ and u is a solution of (1.1.2)*, we obtain, replacing d by φ,*

$$\int_\Omega \varphi^{1-q}|u|^q dx \leq C_{13}\left(1 + \int_\Omega \varphi|\nabla u|^q dx\right) \tag{1.4.22}$$

where $C_{13} = C_{13}(q, \Omega)$.

Proof. Without loss of generality, we can assume that u is nonnegative. By the system of flow coordinates introduced in the section 1.2.1, for any $x \in \Omega_{\delta^*}$, we can write $u(x) = u(\delta, \sigma)$ where $\delta = d(x)$, $\sigma = \sigma(x)$ and $x = \sigma - \delta \mathbf{n}_\sigma$, thus

$$u(\delta,\sigma) - u(\delta^*,\sigma) = -\int_\delta^{\delta^*} \nabla u(\sigma - s\mathbf{n}_\sigma).\mathbf{n}_\sigma ds = -\int_\delta^{\delta^*} \frac{\partial u}{\partial s}(s,\sigma) ds,$$

which follows

$$u(\delta,\sigma) \leq u(\delta^*,\sigma) - \int_\delta^{\delta^*} \frac{\partial u}{\partial s}(s,\sigma) ds.$$

Thus, by multiplying both sides by $(\delta-\gamma)^{-\alpha}$ and integrating on (γ,δ^*), we obtain

$$\begin{aligned}
&\int_{\gamma}^{\delta^*}(\delta-\gamma)^{-\alpha}u(\delta,\sigma)d\delta \\
&\quad\leq \frac{(\delta^*-\gamma)^{1-\alpha}}{1-\alpha}u(\delta^*,\sigma)+\int_{\gamma}^{\delta^*}(\delta-\gamma)^{-\alpha}\int_{\delta}^{\delta^*}|\nabla u(s,\sigma)|\,ds\,d\delta \qquad (1.4.23)\\
&\quad= \frac{(\delta^*-\gamma)^{1-\alpha}}{1-\alpha}u(\delta^*,\sigma)+\frac{1}{1-\alpha}\int_{\gamma}^{\delta^*}(s-\gamma)^{1-\alpha}|\nabla u(s,\sigma)|\,ds.
\end{aligned}$$

Integrating on Σ and using the fact that the mapping is a C^1 diffeomorphism, we get the claim when $q=1$. If $q>1$, we apply (1.4.23) to u^q instead of u and obtain by Holder inequality

$$\begin{aligned}
&\int_{\gamma}^{\delta^*}(\delta-\gamma)^{-\alpha}u^q(\delta,\sigma)d\delta \\
&\leq \frac{(\delta^*-\gamma)^{1-\alpha}}{1-\alpha}u^q(\delta^*,\sigma)+\frac{q}{1-\alpha}\int_{\gamma}^{\delta^*}(s-\gamma)^{1-\alpha}u^{q-1}|\nabla u(s,\sigma)|\,ds \\
&\leq \frac{(\delta^*-\gamma)^{1-\alpha}}{1-\alpha}u^q(\delta^*,\sigma)+\frac{q}{1-\alpha}\left(\int_{\gamma}^{\delta^*}(\delta-\gamma)^{-\alpha}u^q ds\right)^{\frac{1}{q'}}\left(\int_{\gamma}^{\delta^*}(\delta-\gamma)^{q-\alpha}|\nabla u|^q\,ds\right)^{\frac{1}{q}}.
\end{aligned} \qquad (1.4.24)$$

Since the following implication is true

$$(A\geq 0, B\geq 0, M\geq 0, A^q\leq M^q+A^{q-1}B)\Longrightarrow (A\leq M+B)$$

we get

$$\begin{aligned}
&\left(\int_{\gamma}^{\delta^*}(\delta-\gamma)^{-\alpha}u^q(\delta,\sigma)d\delta\right)^{\frac{1}{q}} \\
&\qquad\leq \left(\frac{(\delta^*-\gamma)^{1-\alpha}}{1-\alpha}\right)^{\frac{1}{q}}u^q(\delta^*,\sigma)+\frac{q}{1-\alpha}\left(\int_{\gamma}^{\delta^*}(\delta-\gamma)^{q-\alpha}|\nabla u|^q\,ds\right)^{\frac{1}{q}}.
\end{aligned} \qquad (1.4.25)$$

Inequality (1.4.21) follows as in the case $q=1$. We obtain (1.4.22) with $\gamma=0$, $\alpha=q-1$ and using the fact that $c_{21}^{-1}d\leq\varphi\leq c_{21}\,d$ in Ω with $c_{21}=c_{21}(N)$. □

Theorem 1.4.4 *Assume $q_c\leq q<2$. Let $K\subset\partial\Omega$ be compact such that $C_{\frac{2-q}{q},q'}(K)=0$. Then any positive moderate solution $u\in C^2(\Omega)\cap C(\overline{\Omega}\setminus K)$ of (1.1.2) such that $|\nabla u|\in L^q_d(\Omega)$ which vanishes on $\partial\Omega\setminus K$ is identically zero.*

Proof. Let $\eta\in C^2(\Sigma)$ with value 1 in a neighborhood U_η of K and such that $0\leq\eta\leq 1$, consider $\zeta=\varphi(\mathbb{P}^{\Omega}[1-\eta])^{2q'}$. It is easy to check that ζ is an admissible test function since

$\zeta(x) + |\nabla\zeta(x)| = O(d^{2q'+1}(x))$ in any neighborhood of $\{x \in \partial\Omega : \eta(x) = 1\}$. Then

$$\int_\Omega |\nabla u|^q \zeta dx = \int_\Omega u\Delta\zeta dx = -\int_\Omega \nabla u.\nabla\zeta dx.$$

Next

$$\nabla\zeta = (\mathbb{P}^\Omega[1-\eta])^{2q'}\nabla\varphi - 2q'(\mathbb{P}^\Omega[1-\eta])^{2q'-1}\varphi\nabla\mathbb{P}^\Omega[\eta],$$

thus

$$\begin{aligned}\int_\Omega |\nabla u|^q\zeta dx &= -\int_\Omega (\mathbb{P}^\Omega[1-\eta])^{2q'}\nabla\varphi.\nabla u dx + 2q'\int_\Omega (\mathbb{P}^\Omega[1-\eta])^{2q'-1}\nabla\mathbb{P}^\Omega[\eta].\nabla u\,\varphi dx \\ &= \int_\Omega u\nabla((\mathbb{P}^\Omega[1-\eta])^{2q'}\nabla\varphi)\,dx + 2q'\int_\Omega (\mathbb{P}^\Omega[1-\eta])^{2q'-1}\nabla\mathbb{P}^\Omega[\eta].\nabla u\,\varphi dx.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_\Omega (\lambda u + |\nabla u|^q)\zeta dx \\ &\quad = -2q'\int_\Omega (\mathbb{P}^\Omega[1-\eta])^{2q'-1}u\nabla\varphi.\nabla\mathbb{P}^\Omega[\eta]dx + 2q'\int_\Omega (\mathbb{P}^\Omega[1-\eta])^{2q'-1}\varphi\nabla u.\nabla\mathbb{P}^\Omega[\eta]dx.\end{aligned} \tag{1.4.26}$$

Since $0 \leq \mathbb{P}^\Omega[1-\eta] \leq 1$, $|\nabla\varphi| \leq c_{22}$ in Ω and by Höder inequality,

$$\left|\int_\Omega (\mathbb{P}^\Omega[1-\eta])^{2q'-1}u\,\nabla\varphi.\nabla\mathbb{P}^\Omega[\eta]\,dx\right| \leq c_{22}\left(\int_\Omega \varphi^{1-q}u^q dx\right)^{\frac{1}{q}}\left(\int_\Omega \varphi|\nabla\mathbb{P}^\Omega[\eta]|^{q'}dx\right)^{\frac{1}{q'}}. \tag{1.4.27}$$

Using (1.4.22) and the fact that $|\nabla u| \in L^q_d(\Omega)$, we obtain

$$\left|\int_\Omega (\mathbb{P}^\Omega[1-\eta])^{2q'-1}u\,\nabla\varphi.\nabla\mathbb{P}^\Omega[\eta]\,dx\right| \leq c_{23}\left(1 + \|\nabla u\|^q_{L^q_d(\Omega)}\right)^{\frac{1}{q}}\left(\int_\Omega d|\nabla\mathbb{P}^\Omega[\eta]|^{q'}dx\right)^{\frac{1}{q'}}, \tag{1.4.28}$$

where $c_{23} = c_{23}(N, q, \Omega)$. Using again Hölder inequality, we can estimate the second term on the right-hand side of (1.4.26) as follows

$$\begin{aligned}\int_\Omega (\mathbb{P}^\Omega[1-\eta])^{2q'-1}\varphi\nabla u.\nabla\mathbb{P}^\Omega[\eta]\,dx &\leq \left(\int_\Omega |\nabla u|^q\varphi dx\right)^{\frac{1}{q}}\left(\int_\Omega \varphi|\nabla\mathbb{P}^\Omega[\eta]|^{q'}dx\right)^{\frac{1}{q'}} \\ &\leq c_{21}\,\|\nabla u\|^q_{L^q_d(\Omega)}\left(\int_\Omega d|\nabla\mathbb{P}^\Omega[\eta]|^{q'}dx\right)^{\frac{1}{q'}}.\end{aligned} \tag{1.4.29}$$

Combining (1.4.26), (1.4.28) and (1.4.29) we derive

$$\int_\Omega (|\nabla u|^q + \lambda u)\,\zeta dx \leq c'_{23}\left(1 + \|\nabla u\|^q_{L^q_d(\Omega)}\right)^{\frac{1}{q}}\left(\int_\Omega d|\nabla\mathbb{P}^\Omega[\eta]|^{q'}dx\right)^{\frac{1}{q'}}. \tag{1.4.30}$$

By [36, Propositon 7' and Lemma 4'],

$$\int_\Omega d|\nabla\mathbb{P}^\Omega[\eta]|^{q'}dx \leq c_{24}\,\|\eta\|^{q'}_{W^{1-\frac{2}{q'},q'}(\Sigma)} = c_{24}\,\|\eta\|^{q'}_{W^{\frac{2-q}{q},q'}(\Sigma)}, \tag{1.4.31}$$

which implies

$$\int_{\Omega} (|\nabla u|^q + \lambda u)\, \zeta dx \leq c_{25} \left(1 + \|\nabla u\|^q_{L^q_d(\Omega)}\right)^{\frac{1}{q}} \|\eta\|_{W^{\frac{2-q}{q},q'}(\Sigma)} \tag{1.4.32}$$

where $c_{25} = c_{25}(N, q, \Omega)$. Since $C_{\frac{2-q}{q},q'}(K) = 0$, there exists a sequence of functions $\{\eta_n\}$ in $C^2(\Sigma)$ such that for any n, $0 \leq \eta_n \leq 1$, $\eta_n \equiv 1$ on a neighborhood of K and $\|\eta_n\|_{W^{\frac{2-q}{q},q'}(\Sigma)} \to 0$ and $\|\eta_n\|_{L^1(\Sigma)} \to 0$ as $n \to \infty$. By letting $n \to \infty$ in (1.4.32) with η replaced by η_n and ζ replaced by $\zeta_n := \varphi(\mathbb{P}[1-\eta_n])^{2q'}$, we deduce that $\int_{\Omega} (|\nabla u|^q + \lambda u)\, \varphi dx = 0$ and the conclusion follows. $\square$

1.4.3 Admissible measures

Theorem 1.4.5 *Assume $q_c \leq q < 2$ and let u be a positive moderate solution of* (1.1.2) *with boundary data $\mu \in \mathfrak{M}_+(\partial\Omega)$. Then $\mu(K) = 0$ for any Borel subset $K \subset \partial\Omega$ such that $C_{\frac{2-q}{q},q'}(K) = 0$.*

Proof. Without loss of generality, we can assume that K is compact. We consider test function η as in the proof of Theorem 1.4.4, put $\zeta = (\mathbb{P}^{\Omega}[\eta])^{2q'}\varphi$ and get

$$\int_{\Omega} (|\nabla u|^q \zeta - u\Delta\zeta)\, dx = -\int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\mu. \tag{1.4.33}$$

By Hopf lemma and since $\eta \equiv 1$ on K,

$$-\int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\mu \geq c_{26}\mu(K).$$

Since

$$-\Delta\zeta = \lambda\zeta + 4q'(\mathbb{P}^{\Omega}[1-\eta])^{2q'-1}\nabla\varphi.\nabla\mathbb{P}^{\Omega}[\eta] - 2q'(2q'-1)(\mathbb{P}^{\Omega}[1-\eta])^{2q'-2}\varphi|\nabla\mathbb{P}^{\Omega}[\eta]|^2,$$

we get

$$c_{26}\mu(K) \leq \int_{\Omega} \left((|\nabla u|^q + u\lambda)\zeta + 4q'(\mathbb{P}^{\Omega}[\eta])^{2q'-1} u \nabla\varphi.\nabla\mathbb{P}^{\Omega}[\eta] \right) dx. \tag{1.4.34}$$

Using again the estimates (1.4.28) and (1.4.31), we obtain as in Theorem 1.4.4

$$\left| \int_{\Omega} (\mathbb{P}^{\Omega}[1-\eta])^{2q'-1} u\, \nabla\mathbb{P}^{\Omega}[\eta].\nabla\varphi\, dx \right| \leq c'_{26} \left(1 + \|\nabla u\|^q_{L^q_d(\Omega)}\right)^{\frac{1}{q}} \|\eta\|_{W^{\frac{2-q}{q},q'}(\Sigma)}. \tag{1.4.35}$$

Therefore

$$c_{26}\mu(K) \leq \int_{\Omega} (|\nabla u|^q + u\lambda)\zeta dx + c'_{26} \left(1 + \|\nabla u\|^q_{L^q_d(\Omega)}\right)^{\frac{1}{q}} \|\eta\|_{W^{\frac{2-q}{q},q'}(\Sigma)}. \tag{1.4.36}$$

As in Theorem 1.4.4, since $C_{\frac{2-q}{q},q'}(K) = 0$, there exists a sequence of functions $\{\eta_n\}$ in $C^2(\Sigma)$ such that for any n, $0 \leq \eta_n \leq 1$, $\eta_n \equiv 1$ on a neighborhood of K and $\|\eta_n\|_{W^{\frac{2-q}{q},q'}(\Sigma)} \to 0$ as $n \to 0$. Thus $\|\eta_n\|_{L^1(\Sigma)} \to 0$ and $\zeta_n := (\mathbb{P}^{\Omega}[\eta_n])^{2q'}\varphi \to 0$ a.e. in Ω. Letting $n \to \infty$ in (1.4.36) with η and ζ replaced by η_n and ζ_n respectively and using the dominated convergence theorem, we deduce that $\mu(K) = 0$. $\square$

1.5 Removability in a domain

In this section we assume that Ω is a bounded open domain in $\mathbb{R}^N$ with a C^2 boundary.

1.5.1 General nonlinearity

We consider the following equation

$$\begin{cases} -\Delta u + \tilde{g}(|\nabla u|) = \tilde{\mu} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5.1}$$

where $\tilde{g}$ is a continuous nondecreasing function vanishing at 0 and $\tilde{\mu}$ is a Radon measure. By a solution we mean a function $u \in L^1(\Omega)$ such that $\tilde{g}(|\nabla u|) \in L^1(\Omega)$ satisfying

$$\int_\Omega (-u\Delta\zeta + \tilde{g}(|\nabla u|)\zeta)\, dx = \int_\Omega \zeta d\tilde{\mu} \tag{1.5.2}$$

for all $\zeta \in X(\Omega)$. The integral subcriticality condition on $\tilde{g}$ is the following

$$\int_1^\infty \tilde{g}(s)s^{-\frac{2N-1}{N-1}} ds < \infty \tag{1.5.3}$$

Theorem 1.5.1 *Assume $\tilde{g} \in \mathcal{G}_0$ satisfies* (1.5.3). *Then for any positive bounded Borel measure $\tilde{\mu}$ in Ω there exists a maximal solution $\overline{u}_{\tilde{\mu}}$ of* (1.5.1). *Furthermore, if $\{\mu_n\}$ is a sequence of positive bounded measures in Ω which converges to a bounded measure $\tilde{\mu}$ in the weak sense of measures in Ω and $\{u_{\mu_n}\}$ is a sequence of solutions of* (1.5.1) *with $\tilde{\mu} = \mu_n$, then there exists a subsequence $\{\mu_{n_k}\}$ such that $\{u_{\mu_{n_k}}\}$ converges to a solution $u_{\tilde{\mu}}$ of* (1.5.1) *in $L^1(\Omega)$ and $\{\tilde{g}(|\nabla u_{\mu_{n_k}}|)\}$ converges to $\tilde{g}(|\nabla u_{\tilde{\mu}}|)$ in $L^1(\Omega)$.*

Proof. Since the proof follows the ideas of the one of Theorem 1.2.2, we just indicate the main modifications.
(i) We first consider a sequence of functions $\mu_n \in C_0^\infty(\Omega)$ converging to ν and denote by w_n the solution of

$$\begin{cases} -\Delta w + \tilde{g}(|\nabla(w + \mathbb{G}^\Omega[\mu_n])|) = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5.4}$$

Then $u_n := w_n + \mathbb{G}^\Omega[\mu_n]$ is a approximate solution to (1.5.1).

(ii) The convergence is performed using

$$\left\|\mathbb{G}^\Omega[\tilde{\mu}]\right\|_{L^1(\Omega)} + \left\|\mathbb{G}^\Omega[\tilde{\mu}]\right\|_{M^{\frac{N}{N-2}}(\Omega)} + \left\|\nabla\mathbb{G}^\Omega[\tilde{\mu}]\right\|_{M^{\frac{N}{N-1}}(\Omega)} \le c_1 \|\tilde{\mu}\|_{\mathfrak{M}(\Omega)} \tag{1.5.5}$$

in Proposition 1.2.3.
(iii) For the construction of the maximal solution we consider u_δ solution of

$$\begin{cases} -\Delta u_\delta + \tilde{g}(|\nabla u_\delta|) = \tilde{\mu} & \text{in } \Omega'_\delta \\ u_\delta = \mathbb{G}^\Omega[\tilde{\mu}] & \text{on } \Sigma_\delta. \end{cases} \tag{1.5.6}$$

Then consequently, $0 < \delta < \delta' \Longrightarrow u_\delta \le u_{\delta'}$ in $\Omega'_{\delta'}$ and $u_\delta \downarrow \overline{u}_{\tilde{\mu}}$. Using similar arguments as in the proof of Theorem 1.2.2 we deduce that $\overline{u}_{\tilde{\mu}}$ is the maximal solution of (1.5.1). $\square$

1.5.2 Power nonlinearity

We consider the following equation

$$-\Delta u + |\nabla u|^q = \tilde{\mu} \tag{1.5.7}$$

where $1 < q < 2$. The study on the above equation also leads to a critical value $q^* = \frac{N}{N-1}$. In the subcritical case $1 < q < q^*$, if $\tilde{\mu}$ is a bounded Radon measure, then the problem

$$\begin{cases} -\Delta u + |\nabla u|^q &= \tilde{\mu} \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.5.8}$$

admits a unique solution $u \in L^1(\Omega)$ such that $|\nabla u|^q \in L^1(\Omega)$ (see [4] for solvability of a much more general class of equation). In the contrary, in the supercritical case, an internal singular set can be removable provided that its Bessel capacity is null. More precisely,

Theorem 1.5.2 *Assume $q^* \leq q < 2$ and $K \subset \Omega$ is compact. If $C_{1,q'}(K) = 0$ then any positive solution $u \in C^2(\overline{\Omega} \setminus K)$ of*

$$-\Delta u + |\nabla u|^q = 0 \tag{1.5.9}$$

in $\Omega \setminus K$ satisfying that $\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS$ is bounded remains bounded and can be extended as a solution of the same equation in Ω.

Proof. Let $\eta \in C_c^\infty(\Omega)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in a neighborhood of K. Put $\zeta = 1 - \eta$ and take $\zeta^{q'}$ for test function, then

$$-q' \int_\Omega \zeta^{q'-1} \nabla u . \nabla \eta dx - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS + \int_\Omega \zeta^{q'} |\nabla u|^q dx = 0.$$

Since

$$\left| \int_\Omega \zeta^{q'-1} \nabla u . \nabla \eta dx \right| \leq \left(\int_\Omega \zeta^{q'} |\nabla u|^q dx \right)^{\frac{1}{q}} \left(\int_\Omega |\nabla \eta|^{q'} dx \right)^{\frac{1}{q'}}.$$

Therefore

$$\int_\Omega \zeta^{q'} |\nabla u|^q dx \leq \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS + q' \left(\int_\Omega \zeta^{q'} |\nabla u|^q dx \right)^{\frac{1}{q}} \left(\int_\Omega |\nabla \eta|^{q'} dx \right)^{\frac{1}{q'}},$$

which implies

$$\int_\Omega \zeta^{q'} |\nabla u|^q dx \leq c_{27} \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS + c_{28} \int_\Omega |\nabla \eta|^{q'} dx. \tag{1.5.10}$$

where $c_i = c_i(q)$ with $i = 27, 28$. Since $C_{1,q'}(K) = 0$, there exists a sequence $\{\eta_n\} \subset C_c^\infty(\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood of K and $\|\nabla \eta_n\|_{L^{q'}(\Omega)} \to 0$ as $n \to \infty$. Then the inequality (1.5.10) remains valid with η replaced by η_n and ζ replaced by $\zeta_n = 1 - \eta_n$. Thus, since $\zeta_n \to 1$ a.e. in Ω, we get

$$\int_\Omega |\nabla u|^q dx \leq c_{27} \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS.$$

Hence, from the hypothesis, we deduce that $|\nabla u| \in L^q(\Omega)$.

Next let $\eta \in C_0^\infty(\Omega)$ and η_n as above, then

$$\int_\Omega (1-\eta_n)\nabla\eta.\nabla u dx - \int_\Omega \eta\nabla\eta_n.\nabla u dx + \int_\Omega (1-\eta_n)\eta|\nabla u|^q dx = 0.$$

Since $|\nabla u| \in L^q(\Omega)$, we can let $n \to \infty$ and obtain by monotone and dominated convergence

$$\int_\Omega (\nabla\eta.\nabla u + \eta|\nabla u|^q)\, dx = 0.$$

Regularity results imply that $u \in C^2(\Omega)$. □

Theorem 1.5.3 *Assume $q^* \leq q < 2$ and $\tilde{\mu} \in \mathfrak{M}_+(\Omega)$. Let $u \in L^1(\Omega)$ with $|\nabla u| \in L^q(\Omega)$ is a solution of* (1.5.8) *in Ω. If $E \subset \Omega$ is a Borel subset satisfying $C_{1,q'}(E) = 0$ then $\tilde{\mu}(E) = 0$.*

Proof. Since $\tilde{\mu}$ is outer regular, it is sufficient to prove the result when E is compact. Let η_n be a sequence as in the previous theorem, then

$$\int_\Omega (\nabla u.\nabla\eta_n + \eta_n|\nabla u|^q)dx = \int_\Omega \eta_n d\tilde{\mu} \geq \tilde{\mu}(E). \tag{1.5.11}$$

But the left-hand side of (1.5.11) is dominated by

$$\left(\int_\Omega |\nabla\eta_n|^{q'} dx\right)^{\frac{1}{q'}} \left(\int_\Omega \eta_n|\nabla u|^q dx\right)^{\frac{1}{q}} + \int_\Omega \eta_n|\nabla u|^q dx,$$

which goes to 0 when $n \to \infty$, both by the definition of the $C_{1,q'}$-capacity and the fact that $\eta_n \to 0$ a.e. as $n \to \infty$ and is bounded by 1. Thus $\tilde{\mu}(E) = 0$. □

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Chapitre 2

Local and global properties of solutions of heat equation with superlinear absorption

Abstract

In this chapter [1], we study the limit, when $k \to \infty$ of the solutions of $\partial_t u - \Delta u + f(u) = 0$ in $\mathbb{R}^N \times (0,\infty)$ with initial data $k\delta_0$, where δ_0 is the Dirac mass concentrated at the origin and f is a positive, superlinear, continuous, increasing function. We prove that there exist essentially three types of possible behaviour according f^{-1} and $F^{-1/2}$ belong or not to $L^1(1,\infty)$, where $F(t) = \int_0^t f(s)ds$. We use these results for providing a new and more general construction of the initial trace and some uniqueness and non-uniqueness results for solutions with unbounded initial data.

[1]This chapter is based on the paper : P. T. Nguyen and L. Véron, *Local and global properties of solutions of heat equation with superlinear absorption*, **Adv. Diff. Equ. 16**, 487-522 (2011).

2.1 Introduction

We investigate some local and global properties of solutions of a class of semilinear heat equations

$$\partial_t u - \Delta u + f(u) = 0 \tag{2.1.1}$$

in $Q_\infty := \mathbb{R}^N \times (0,\infty)$ $(N \geq 2)$ where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nondecreasing, positive on $(0,\infty)$, vanishes at 0 and satisfies $\lim_{s\to\infty} f(s) = \infty$. As a *model equation* we shall consider the following nonlinear term, with $\alpha > 0$,

$$\partial_t u - \Delta u + u \ln^\alpha(u+1) = 0, \tag{2.1.2}$$

which points out all the delicate features of *weakly superlinear absorption*. By opposition, for power-like absorption $f(u) = |u|^\beta u$ with $\beta > 0$ much is known about the structure of the set of solutions. The local and asymptotic behaviour of solutions is strongly linked to the existence of a self-similar solutions under the form

$$u(x,t) = t^{\frac{-1}{\beta}} \Psi\left(\frac{x}{\sqrt{t}}\right) \tag{2.1.3}$$

where Ψ is the solution of a ordinary differential equation (see [1], [2]). In this case the critical exponent $\beta_c = \frac{2}{N}$ plays a fundamental role in the description of isolated singularities and the study of the initial trace. This is due to the fact that, for $0 < \beta < \beta_c$, there exists a positive self-similar solution with an isolated singularity at $(0,0)$ and vanishing on $\mathbb{R}^N \times \{0\} \setminus \{(0,0)\}$, while no such solution exists when $\beta \geq \beta_c$ and more generally, no solution with isolated singularities.

In the case of (2.1.2), no self-similar structure exists. There is no critical exponent corresponding to isolated singularities since there always exist such singular solutions. Actually, for any $k > 0$ there exists a unique $u = u_{k\delta_0} \in C(\overline{Q_\infty} \setminus \{(0,0)\}) \cap C^{2,1}(Q_\infty)$ solution of

$$\left\{\begin{array}{ll} \partial_t u - \Delta u + u \ln^\alpha(u+1) = 0 & \text{in } Q_\infty \\ u(.,0) = k\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N). \end{array}\right. \tag{2.1.4}$$

There are two critical values for $\alpha : \alpha = 1$ and $\alpha = 2$, the explanation of which comes from the study of the two singular problems

$$\left\{\begin{array}{lll} \phi' + \phi \ln^\alpha(\phi+1) & = 0 & \text{in } (0,\infty) \\ \phi(0) & = \infty, & \end{array}\right. \tag{2.1.5}$$

and, for any $\epsilon > 0$,

$$\left\{\begin{array}{lll} -\Delta\psi + \psi \ln^\alpha(\psi+1) & = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim\limits_{|x|\to\epsilon} \psi(x) & = \infty, & \end{array}\right. \tag{2.1.6}$$

where $B_\epsilon := \{x \in \mathbb{R}^N : |x| < \epsilon\}$. When it exists, the solution ϕ_∞ of (2.1.5) is given implicitely by

$$\int_{\phi_\infty(t)}^\infty \frac{ds}{s \ln^\alpha(s+1)} = t \qquad \forall t > 0, \tag{2.1.7}$$

and such a formula is valid if and only if $\alpha > 1$. For problem (2.1.6) an explicit expression of the solution is not valid, but this solution exists if and only if $\alpha > 2$; in this case, the Keller-Osserman condition (see (2.1.12) below) holds.

Having in mind this model we study (2.1.1) assuming the *weak singularity condition* on f :

$$\int_1^\infty s^{-2-\frac{2}{N}} f(s)ds < \infty. \tag{2.1.8}$$

Proposition 2.1.1 *Assume* (2.1.8) *holds. Then for any $k > 0$, there exists a unique solution $u := u_{k\delta_0}$ to*

$$\begin{cases} \partial_t u - \Delta u + f(u) &= 0 \quad \text{in } Q_\infty \\ u(.,0) &= k\delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \tag{2.1.9}$$

Furthermore, if ψ_n is a sequence of positive integrable functions converging to $k\delta_0$ in the weak-star topology, then the sequence u_{ψ_n} of solutions of (2.1.1) *in Q_∞ with initial data ψ_n converges to $u_{k\delta_0}$, locally uniformly.*

Another important condition on f is

$$\int_1^\infty \frac{ds}{f(s)} < \infty. \tag{2.1.10}$$

Under assumption (2.1.10) there exists a maximal solution $\phi := \phi_\infty$ of

$$\phi' + f(\phi) = 0 \qquad \text{in } (0, \infty) \tag{2.1.11}$$

which satisfies $\lim_{t\to 0} \phi_\infty(t) = \infty$. This function is explicited by a formula similar to (2.1.7) in which $s \ln^\alpha(s + 1)$ is replaced by $f(s)$.

The next important condition on f we shall encounter is the Keller-Osserman condition, i.e.

$$\int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \tag{2.1.12}$$

where

$$F(s) = \int_0^s f(r)dr, \quad \forall s \in [1, \infty). \tag{2.1.13}$$

If (2.1.12) is satisfied, by [6, Theorem III] for any $\epsilon > 0$ there exists a maximal solution $\psi := \psi_\epsilon$ of

$$-\Delta\psi + f(\psi) = 0 \qquad \text{in } \mathbb{R}^N \setminus B_\epsilon \tag{2.1.14}$$

which satisfies $\lim_{|x|\to\epsilon} \psi(x) = \infty$. Assumptions (2.1.10) and (2.1.13) which are simultaneously satisfied in the case of a power like absorption, but not in our model case. This illuminates the structure of the set of solutions of (2.1.1), in particular in view of the initial trace problem.

The first question we consider is the study of the limit of $u_{k\delta_0}$ when $k \to \infty$. This question is natural since $k \mapsto u_{k\delta_0}$ is increasing. In order to treat it, we need the super-additivity on f, i.e.

$$f(s + s') \geq f(s) + f(s') \quad \forall s, s' \geq 0. \tag{2.1.15}$$

From this condition and the monotonicity of f, we deduce a minimal linear growth at infinity

$$\liminf_{s\to\infty} \frac{f(s)}{s} > 0. \tag{2.1.16}$$

Moreover, notice that if f satisfies (2.1.12) and (2.1.16) then (2.1.10) holds.

In the second section, we prove the following results.

Theorem 2.1.2 *Assume $f(s) = s\ln^\alpha(s+1)$ avec $0 < \alpha \leq 1$. Then the solutions $u_{k\delta_0}$ of* (2.1.4) *satisfy $\lim_{k\to\infty} u_{k\delta_0}(x,t) = \infty$ for every $(x,t) \in Q_\infty$.*

Theorem 2.1.3 *Assume $f(s) = s\ln^\alpha(s+1)$ avec $1 < \alpha \leq 2$. Then the solutions $u_{k\delta_0}$ of* (2.1.4) *satisfy $\lim_{k\to\infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ for every $(x,t) \in Q_\infty$, where ϕ_∞ is the solution of* (2.1.5).

We denote by $\mathcal{U}_0$ the set of positive solutions u of (2.1.1) in Q_∞, which are continuous in $\overline{Q_\infty} \setminus \{(0,0)\}$, vanish on the set $\{(x,0) : x \neq 0\}$ and satisfies

$$\lim_{t\to 0} \int_{B_\epsilon} u(x,t)dx = \infty \tag{2.1.17}$$

for any $\epsilon > 0$.

If $f(s) = s\ln^\alpha(s+1)$ with $\alpha > 2$, we obtain a result of minimal element of $\mathcal{U}_0$ which comes from the following theorem :

Theorem 2.1.4 *Assume f satisfies* (2.1.8), (2.1.12) *and* (2.1.15). *Then $\underline{U} := \lim_{k\to\infty} u_{k\delta_0}$ is the minimal element of $\mathcal{U}_0$, where $u_{k\delta_0}$ is the solution of* (2.1.9).

In the third section we study the set of positive and locally bounded solutions of (2.1.1) in Q_∞. This set differs considerably according the assumption on f. This is due to the properties of the radial solutions of the associated stationnary equation

$$-\Delta w + f(w) = 0 \qquad \text{in } \mathbb{R}^N. \tag{2.1.18}$$

The next result is based upon the Picard-Lipschitz fixed point theorem and a result of Vázquez and Véron [13].

Proposition 2.1.5 *Assume f is locally Lipschitz continuous on $\mathbb{R}_+$ and* (2.1.12) *does not holds. For any $a > 0$, there exists a unique positive function $w := w_a \in C^2([0,\infty))$ to the problem*

$$\begin{cases} -w'' - \dfrac{N-1}{r}w' + f(w) &= 0 \qquad \text{in } \mathbb{R}_+ \\ w'(0) = 0, \qquad w(0) &= a. \end{cases} \tag{2.1.19}$$

A striking consequence of the existence of such solutions is the following non-uniqueness result.

Theorem 2.1.6 *Assume f is locally Lipschitz continuous on $\mathbb{R}_+$ and satisfies* (2.1.10) *but does not satisfy* (2.1.12). *Then for any $u_0 \in C(\mathbb{R}^N)$ satisfying, for some $b > a > 0$, $w_a(x) \leq u_0(x) \leq w_b(x)$ $\forall x \in \mathbb{R}^N$, there exist two solutions $\underline{u}, \overline{u} \in C(\overline{Q_\infty})$ of* (2.1.1) *with initial value u_0. They satisfy respectively*

$$0 \leq \underline{u}(x,t) \leq \min\{w_b(x), \phi_\infty(t)\} \qquad \forall (x,t) \in Q_\infty, \tag{2.1.20}$$

thus $\lim_{t\to\infty} \underline{u}(x,t) = 0$, uniformly with respect to $x \in \mathbb{R}^N$, and

$$w_a(x) \leq \overline{u}(x,t) \leq w_b(x) \qquad \forall (x,t) \in Q_\infty, \tag{2.1.21}$$

thus $\lim_{|x|\to\infty} \overline{u}(x,t) = \infty$, uniformly with respect to $t \geq 0$.

The next theorem shows that if two solutions of (2.1.1) have the same initial data and the same asymptotic behaviour as $|x| \to \infty$ then they coincide.

Theorem 2.1.7 *Assume f is locally Lipschitz continuous on $\mathbb{R}_+$ and satisfies* (2.1.15) *but does not satisfy* (2.1.12). *Let u and $\tilde{u}$ be two positive solutions in $C(\overline{Q_\infty}) \cap C^{2,1}(Q_\infty)$ of* (2.1.1) *with initial data $u_0 \in C(\mathbb{R}^N)$. If for any $\epsilon > 0$,*

$$u(x,t) - \tilde{u}(x,t) = o(w_\epsilon(|x|)) \text{ as } x \to \infty \tag{2.1.22}$$

locally uniformly with respect to $t \geq 0$, then $u = \tilde{u}$.

On the contrary, if the Keller-Osserman condition (2.1.12) holds, a continuous solution is uniquely determined by the positive initial value $u_0 \in C(\mathbb{R}^N)$, and uniqueness still holds if $C(\mathbb{R}^N)$ is replaced by $\mathfrak{M}_+(\mathbb{R}^N)$.

Theorem 2.1.8 *Assume f satisfies* (2.1.12) *and* (2.1.15). *Then*

(i) For any nonnegative function $u_0 \in C(\mathbb{R}^N)$ there exists a unique nonnegative solution $u \in C(\overline{Q_\infty})$ of (2.1.1) *in Q_∞ with initial value u_0.*

(ii) For any for any nonnegative measure $\mu \in \mathfrak{M}(\mathbb{R}^N)$, there exists at most one nonnegative solution $u \in C(Q_\infty)$ of (2.1.1) *in Q_∞ such that $f(u) \in L^1_{loc}(\overline{Q_\infty})$ satisfying*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N} \zeta(x)d\mu(x) \quad \forall \zeta \in C_c(\mathbb{R}^N). \tag{2.1.23}$$

In the last section we use the tools studied in the previous sections to develop a *new construction of the initial trace* of locally bounded positive solutions of (2.1.1) in Q_∞. By opposition to the power-like case [7], where the initial trace was constructed by duality arguments based upon Hölder inequality and delicate choice of test functions, our new method has the advantage of being based only on maximum principle, using either the Keller-Osserman condition (2.1.12), or the asymptotics of the $u_{k\delta_0}$ if (2.1.12) does not hold where $u_{k\delta_0}$ is the solution of (2.1.9). We first prove

Proposition 2.1.9 *Let $u \in C^{2,1}(Q_\infty)$ be a positive solution of* (2.1.1) *in Q_∞. The set $\mathcal{R}(u)$ of the points $z \in \mathbb{R}^N$ such that there exists an open ball $B_r(z)$ such that $f(u) \in L^1(Q_T^{B_r(z)})$ is an open subset. Furthermore there exists a positive Radon measure $\mu := \mu(u)$ on $\mathcal{R}(u)$ such that*

$$\lim_{t\to 0} \int_{\mathcal{R}(u)} u(x,t)\zeta(x)dx = \int_{\mathcal{R}(u)} \zeta(x)d\mu(x) \qquad \forall \zeta \in C_c(\mathcal{R}(u)). \tag{2.1.24}$$

Due to Proposition 2.1.9, we introduce the definition of the initial trace.

Definition 2.1.10 *The couple $(\mathcal{S}(u), \mu)$ where $\mathcal{S}(u) = \mathbb{R}^N \setminus \mathcal{R}(u)$ is called the initial trace of u in Ω and will be denoted by $tr_{\mathbb{R}^N}(u)$. The set $\mathcal{R}(u)$ is called the regular set of the initial trace of u and the measure μ is called the regular part of the initial trace. The set $\mathcal{S}(u)$ is closed and is called the singular part of the initial trace of u.*

The initial trace can also be represented by a positive, outer regular Borel measure, not necessary locally bounded. The space of these measures on $\mathbb{R}^N$ will be denoted by $\mathcal{B}_+^{reg}(\mathbb{R}^N)$. There is a one-to-one correspondence between $\mathcal{B}_+^{reg}(\mathbb{R}^N)$ and the set of couples :

$$CM_+(\mathbb{R}^N) = \left\{(\mathcal{S}, \mu) : \mathcal{S} \subset \mathbb{R}^N \text{ closed}, \mu \in \mathfrak{M}_+(\mathcal{R}) \text{ with } \mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}\right\}. \tag{2.1.25}$$

The Borel measure $\nu \in \mathcal{B}_+^{reg}(\mathbb{R}^N)$ corresponding to a couples $(\mathcal{S}, \mu) \in CM_+(\mathbb{R}^N)$ is given by

$$\nu(A) = \begin{cases} \infty & \text{if } A \cap \mathcal{S} \neq \emptyset \\ \mu(A) & \text{if } A \subseteq \mathcal{R}, \end{cases} \qquad \forall A \subset \mathbb{R}^N, A \text{ Borel}. \tag{2.1.26}$$

If u is a solution of (2.1.1), we shall use the notation $tr_{\mathbb{R}^N}(u)$ (resp. $Tr_{\mathbb{R}^N}(u)$) for the trace considered as an element of $CM_+(\mathbb{R}^N)$ (resp. $\mathcal{B}_+^{reg}(\mathbb{R}^N)$).

We consider the case when the Keller-Osserman holds.

Theorem 2.1.11 *Assume f is nondecreasing and satisfies* (2.1.12). *If $u \in C^{2,1}(Q_\infty)$ is a positive solution of* (2.1.1), *it possesses an initial trace $\nu \in \mathcal{B}_+^{reg}(\mathbb{R}^N)$.*

Furthermore, the following theorem deals with the existence of the maximal solution and the minimal solution of (2.1.1) with a given initial trace $(\mathcal{S}, \mu) \in CM_+(\mathbb{R}^N)$.

Theorem 2.1.12 *Assume f is nondecreasing and satisfies* (2.1.12), (2.1.8) *and* (2.1.15). *Then for any $(\mathcal{S}, \mu) \in CM_+(\mathbb{R}^N)$ there exist a maximal solution $\overline{u}_{\mathcal{S},\mu}$ and a minimal solution $\underline{u}_{\mathcal{S},\mu}$ of* (2.1.1) *in Q_∞, with initial trace $(\mathcal{S}, \mu)$, in the following sense :*

$$\underline{u}_{\mathcal{S},\mu} \le v \le \overline{u}_{\mathcal{S},\mu} \tag{2.1.27}$$

for every positive solution $v \in C^{2,1}(Q_\infty)$ of (2.1.1) *in Q_∞ such that $tr_{\mathbb{R}^N}(v) = (\mathcal{S}, \mu)$.*

If the Keller-Osserman does not hold, we obtain the following results which depend upon $\lim_{k\to\infty} u_{k\delta_0}$ is equal to ϕ_∞ or is infinite (we recall that $u_{k\delta_0}$ is the solution of (2.1.9)).

Theorem 2.1.13 *Assume* (2.1.8), (2.1.10) *are verified but* (2.1.12) *does not hold. Moreover suppose that* $\lim_{k\to\infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ *for every* $(x,t) \in Q_\infty$. *If* u *is a positive solution of* (2.1.1) *in* Q_∞, *it possesses an initial trace which is either the Borel measure* ν_∞ *which satisfies* $\nu_\infty(\mathcal{O}) = \infty$ *for any non-empty open subset* $\mathcal{O} \subset \mathbb{R}^N$, *or is a positive Radon measure* μ *on* $\mathbb{R}^N$. *This result holds in particular if* $f(s) = s\ln^\alpha(s+1)$ *with* $1 < \alpha \leq 2$.

A consequence of Theorem 2.1.13 which is worth mentioning is the following :

Proposition 2.1.14 *Under the assumptions of Theorem 2.1.13 and the condition* (2.1.15), *for any* $b > 0$ *there exists a positive solution* $u \in C(Q_\infty)$ *of* (2.1.1) *satisfying*

$$\max\{\phi_\infty(t); w_b(|x|)\} \leq u(x,t) \leq \phi_\infty(t) + w_b(|x|) \qquad \forall (x,t) \in Q_\infty. \tag{2.1.28}$$

Consequently there exist infinitely many positive solutions of (2.1.1) *with initial trace* ν_∞. *Furthermore* ϕ_∞ *is the smallest of all these solutions.*

Theorem 2.1.15 *Assume* f *satisfies* (2.1.8) *but neither* (2.1.10) *nor* (2.1.12). *Moreover suppose that* $\lim_{k\to\infty} u_{k\delta_0} = \infty$ *in* Q_∞. *If* u *is a positive solution of* (2.1.1) *in* Q_∞, *it possesses an initial trace which is a positive Radon measure* μ *on* $\mathbb{R}^N$. *This result holds in particular if* $f(s) = s\ln^\alpha(s+1)$ *with* $0 < \alpha \leq 1$.

The proofs are combination of methods developed in [10] for elliptic equations, stability results and Theorem 2.1.2 and Theorem 2.1.3.

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2.2 Isolated singularities

In order to study (2.1.1), we start proving Proposition 2.1.1.

Proof of Proposition 2.1.1. We denote by $E(x,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$ the fundamental solution of the heat equation in Q_∞. Since kE $(k > 0)$ is a supersolution for (2.1.1), it is classical to prove that if

$$I := \int_0^1 \int_{B_R} f(kE(x,t))dx\,dt < \infty \tag{2.2.1}$$

for any $R > 0$, then there exists a unique solution $u = u_{k\delta_0}$ of (2.1.1) satisfying initial condition $u_{k\delta_0}(.,0) = k\delta_0$ in $\mathcal{D}'(\mathbb{R}^N)$. Furthermore the mapping $k \mapsto u_{k\delta_0}$ is increasing. Actually, it is proved in [8, Th 1.1] that if f satisfies the weak singularity assumption (2.1.8), then for any positive bounded Borel measure there exists a unique solution $u := u_\mu$ of (2.1.1) satisfying $u_\mu(.,0) = \mu$. Furthermore if $\{\mu_n\}$ is a sequence of positive bounded measures which converge to a measure μ in the weak-star topology of measures, then the sequence of corresponding solutions $\{u_{\mu_n}\}$ converges locally uniformly to u_μ, and $\{f(u_{\mu_n})\}$ converges to $f(u_\mu)$ in $L^1_{loc}(\mathbb{R}^N \times [0,\infty))$.

This existence result and the next proposition lead to the conclusion of Proposition 2.1.1. $\square$

Proposition 2.2.1 *If f satisfies* (2.1.8) *then* (2.2.1) *is fulfilled.*

Proof. In this proof, c denote a constant depending on N whose value may change line by line. By linearity we can assume that $k = (4\pi)^{\frac{N}{2}}$. Moreover assume that $R = 1$, then

$$I = \int\int_{B_1\times(0,1)} f(v_k)dx\,dt = \omega_N \int_0^1\int_0^1 f\left(t^{-\frac{N}{2}}e^{-\frac{r^2}{4t}}\right) r^{N-1}dr\,dt.$$

Set $s = t^{-\frac{N}{2}}e^{-\frac{r^2}{4t}}$, then

$$\begin{aligned} I &= 2^{N-1}\omega_N \int_0^1\int_{t^{-\frac{N}{2}}e^{-\frac{1}{4t}}}^{t^{-\frac{N}{2}}} \left[-\ln s - \ln\left(t^{\frac{N}{2}}\right)\right]^{\frac{N-2}{2}} f(s)s^{-1}ds\,t^{\frac{N}{2}}dt \\ &\le 2^{N-1}\omega_N \int_0^1\int_{e^{-\frac{1}{4t}}}^{t^{-\frac{N}{2}}} \left[-\ln s - \ln\left(t^{\frac{N}{2}}\right)\right]^{\frac{N-2}{2}} f(s)s^{-1}ds\,t^{\frac{N}{2}}dt \quad \le 2^{N-1}\omega_N(I_1 + I_2) \end{aligned} \tag{2.2.2}$$

where,

$$\begin{aligned} I_1 &= \int_0^{e^{-\frac{1}{4}}}\int_0^{-\frac{1}{4\ln s}} \left[-\ln s - \ln\left(t^{\frac{N}{2}}\right)\right]^{\frac{N-2}{2}} t^{\frac{N}{2}}dt\,s^{-1}f(s)ds \\ &= \frac{2}{N}\int_0^{e^{-\frac{1}{4}}}\int_0^{\frac{s}{(-4\ln s)^{\frac{N}{2}}}} (-\ln\tau)^{\frac{N-2}{2}}\tau^{\frac{2}{N}}d\tau\,s^{-2-\frac{2}{N}}f(s)ds, \end{aligned}$$

by setting $\tau = st^{\frac{N}{2}}$. But

$$\begin{aligned} \int_0^{\frac{s}{(-4\ln s)^{\frac{N}{2}}}} (-\ln\tau)^{\frac{N-2}{2}}\tau^{\frac{2}{N}}d\tau &\le c\left[(-\ln\tau)^{\frac{N-2}{2}}\tau^{\frac{N+2}{N}}\right]_0^{\frac{s}{(-4\ln s)^{\frac{N}{2}}}} \\ &\le cs^{1+\frac{2}{N}}(-\ln s)^{-2}\left(1 + \frac{N}{2}\frac{\ln(-4\ln s)}{-\ln s}\right)^{\frac{N-2}{2}} \\ &\le cs^{1+\frac{2}{N}}(-\ln s)^{-2}, \end{aligned}$$

thus

$$I_1 \le c\int_0^{e^{-\frac{1}{4}}} s^{-1}(-\ln s)^{-2}f(s)ds < \infty \tag{2.2.3}$$

by Duhamel's rule. Further

$$\begin{aligned} I_2 &\le \int_{e^{-\frac{1}{4}}}^{\infty}\int_0^{s^{-\frac{2}{N}}} \left[-\ln s - \ln\left(t^{\frac{N}{2}}\right)\right]^{\frac{N-2}{2}} t^{\frac{N}{2}}dt\,s^{-1}f(s)ds \\ &\le \frac{2}{N}\int_{e^{-\frac{1}{4}}}^{\infty}\int_0^1 (-\ln\tau)^{\frac{N-2}{2}}\tau^{\frac{2}{N}}d\tau\,s^{-2-\frac{2}{N}}f(s)ds \\ &\le c\int_{e^{-\frac{1}{4}}}^{\infty} s^{-2-\frac{2}{N}}f(s)ds. \end{aligned} \tag{2.2.4}$$

The conclusion follows from (2.2.2)–(2.2.4). □

For $a > 0$, we denote by ϕ_a the solution of (2.1.11) with initial data $\phi(0) = a$. If (2.1.10) does not hold then $\lim_{a\to\infty} \phi_a(t) = \infty$ for any $t \in (0, \infty)$. While, if (2.1.10) holds, the solution $\phi_\infty = \lim_{a\to\infty} \phi_a$ and it is given explicitely by

$$t = \int_{\phi_\infty(t)}^{\infty} \frac{ds}{f(s)} < \infty.$$

Proof of Theorem 2.1.2. For any $k > 0$, since kE is a super-solution of (2.1.2), it follows from the maximum principle that $u_{k\delta_0} \le kE$ in Q_∞, which implies $u_{k\delta_0}(x,t) \le kC^* t^{-\frac{N}{2}}$ for every $(x,t) \in Q_\infty$ where $C^* = (4\pi)^{-\frac{N}{2}}$. Thus

$$\partial_t u_{k\delta_0} - \Delta u_{k\delta_0} + u_{k\delta_0} \ln^\alpha(1 + kC^* t^{-\frac{N}{2}}) \ge 0.$$

Set $\theta(t) = \ln^\alpha(1 + kC^* t^{-\frac{N}{2}})$, $\Theta(t) = \int_0^t \theta(s)ds$ and $v_{k\delta_0}(x,t) = e^{\Theta(t)} u_{k\delta_0}(x,t)$. Then

$$\partial_t v_{k\delta_0} - \Delta v_{k\delta_0} = e^{\Theta(t)}(\partial_t u_{k\delta_0} - \Delta u_{k\delta_0} + u_{k\delta_0}\theta(t)) \ge 0,$$

and $v_{k\delta_0}(.,0) = u_{k\delta_0}(.,0) = k\delta_0$. By maximum principle, it follows that

$$v_{k\delta_0}(x,t) \ge kC^* t^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} \Longleftrightarrow u_{k\delta_0}(x,t) \ge kC^* t^{-\frac{N}{2}} e^{-\Theta(t) - \frac{|x|^2}{4t}}. \tag{2.2.5}$$

Next, if we restrict to $0 < t \le 1$, k large enough, put $a = \frac{N}{2}\ln(t^{-1})$, $b = \ln(t^{\frac{N}{2}} + kC^*)$, and apply the following inequality

$$(a + b)^\alpha \le a^\alpha + b^\alpha \quad \forall 0 < \alpha \le 1 \tag{2.2.6}$$

in order to obtain

$$\begin{aligned} \theta(t) = \ln^\alpha(1 + kC^* t^{-\frac{N}{2}}) &= (\tfrac{N}{2}\ln(t^{-1}) + \ln(t^{\frac{N}{2}} + kC^*))^\alpha \\ &\le \tfrac{N^\alpha}{2^\alpha}\ln^\alpha(t^{-1}) + \ln^\alpha(t^{\frac{N}{2}} + kC^*) \\ &\le \tfrac{N^\alpha}{2^\alpha}\ln^\alpha(t^{-1}) + \ln^\alpha k. \end{aligned} \tag{2.2.7}$$

Therefore, we always assume that $0 < t \le 1$, and get

$$\Theta(t) \le \int_0^1 \frac{N^\alpha}{2^\alpha}\ln^\alpha(t^{-1})dt + t\ln^\alpha k,$$

which follows that

$$e^{-\Theta(t)} \ge c_1 e^{-t\ln^\alpha k}$$

where

$$c_1 = \exp\left(-\frac{N^\alpha}{2^\alpha}\int_0^1 \ln^\alpha(t^{-1})dt\right).$$

Therefore

$$u_{k\delta_0}(x,t) \ge c_1 C^* t^{-\frac{N}{2}} e^{\ln k - t\ln^\alpha k - \frac{|x|^2}{4t}} \quad \forall (x,t) \in \mathbb{R}^N \times (0,1].$$

Since $0 < \alpha \le 1$, $\ln k - t\ln^\alpha k \to \infty$ as $k \to \infty$. Hence

$$\lim_{k\to\infty} u_{k\delta_0}(x,t) = \infty$$

uniformly in every compact subset of $\mathbb{R}^N \times (0,1)$. This implies the claim. □

Proposition 2.2.2 *Assume* (2.1.10) *is satisfied. For any* $k > 0$*, there holds*

$$u_{k\delta_0}(x,t) \leq \phi_\infty(t) \qquad \forall (x,t) \in Q_\infty.$$

Proof. For any small $\epsilon > 0$, we set $\phi_{\infty\epsilon}(t) = \phi_\infty(t-\epsilon), t \in [\epsilon, \infty)$ then $\phi_{\infty\epsilon}$ is a solution of (2.1.1) in (ϵ, ∞), which dominates $u_{k\delta_0}$ on $\mathbb{R}^N \times \{\epsilon\}$ for any $k > 0$. By comparison principle, $u_{k\delta_0}(x,t) \leq \phi_{\infty\epsilon}(t)$ for every $(x,t) \in \mathbb{R}^N \times [\epsilon, \infty)$. Letting $\epsilon \to 0$ yields the claim. □

Proof of Theorem 2.1.3. By using the same argument as in the proof of Theorem 2.1.2 and employing the following inequality

$$(a+b)^\alpha \leq 2^{\alpha-1}(a^\alpha + b^\alpha) \quad \forall \alpha \geq 1$$

instead of (2.2.6), we obtain

$$u_{k\delta_0}(x,t) \geq C^* t^{-\frac{N}{2}} e^{\ln k - 2^{\alpha-1} t \ln^\alpha k - \frac{|x|^2}{4t}} \quad \forall (x,t) \in \mathbb{R}^N \times (0,1].$$

The sequence $\{u_{k\delta_0}\}$ is increasing and bounded from above by ϕ_∞, then there exists $\underline{U} = \lim_{k\to\infty} u_{k\delta_0}$. Since $\underline{U} \geq u_{k\delta_0}$,

$$\underline{U}(x,t) \geq C^* t^{-\frac{N}{2}} e^{\ln k - 2^{\alpha-1} t (\ln k)^\alpha - \frac{|x|^2}{4t}} \quad \forall (x,t) \in \mathbb{R}^N \times (0,1], \forall k > 0 \tag{2.2.8}$$

Let $\{t_n\} \subset (0,1]$ be a sequence converging to 0. We choose $k = k_n = \exp(2^{\frac{\alpha}{1-\alpha}} t_n^{\frac{1}{1-\alpha}})$, then $\ln k_n - 2^{\alpha-1} t_n \ln^\alpha k_n = 2^{\frac{2\alpha-1}{1-\alpha}} t_n^{\frac{1}{1-\alpha}}$. Next we restrict x in order

$$\ln t_n - 2^{\alpha-1} t_n \ln^\alpha t_n - \frac{|x|^2}{4t_n} = 2^{\frac{2\alpha-1}{1-\alpha}} t_n^{\frac{1}{1-\alpha}} - \frac{|x|^2}{4t_n} \geq 0 \Longleftrightarrow |x| \leq 2^{\frac{1}{2(1-\alpha)}} t_n^{\frac{2-\alpha}{2(1-\alpha)}}.$$

Therefore, since $1 < \alpha \leq 2$,

$$\lim_{n\to\infty} \underline{U}(x, t_n) = \infty$$

uniformly on $\mathbb{R}^N$ if $1 \leq \alpha < 2$, or uniformly on the ball $B_{2^{-1/2}}$ if $\alpha = 2$. Since the sequence $\{t_n\}$ is arbitrary,

$$\lim_{t\to 0} \underline{U}(x,t) = \infty$$

uniformly on $\mathbb{R}^N$ if $1 \leq \alpha < 2$, or uniformly on the ball $B_{2^{-1/2}}$ if $\alpha = 2$.

We pick some point x_0 in $\mathbb{R}^N$ (resp. $B_{2^{-1/2}}$) if $1 < \alpha < 2$ (resp. $\alpha = 2$). Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (2.1.2) with initial data $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)(0 < \sigma < 2^{-1/2} - |x_0|)$, by comparison principle, it follows that

$$\underline{U}(x,t) \geq u_{k\delta_{x_0}}(x,t) = u_{k\delta_0}(x - x_0, t).$$

Letting $k \to \infty$ yields to $\underline{U}(x,t) \geq \underline{U}(x - x_0, t)$. Reversing the role of 0 and x_0 yields to $\underline{U}(x,t) = \underline{U}(x - x_0, t)$. If we iterate this process we derive

$$\underline{U}(x,t) = \underline{U}(x - y, t), \quad \forall y \in \mathbb{R}^N.$$

This implies that $\underline{U}$ is independent of x and therefore it is the solution of (2.1.5). Thus $\underline{U} = \phi_\infty$. □

Proposition 2.2.3 *Assume* (2.1.12) *and* (2.1.8) *are satisfied. Then for any* $k > 0$ *there holds*

$$u_{k\delta_0}(x,t) \leq \Phi(|x|) \quad \forall (x,t) \in Q_\infty$$

where Φ *is a solution to the problem*

$$\begin{cases} -\Phi'' + f(\Phi) &= 0 \qquad \text{in } (0,\infty) \\ \lim\limits_{s\to 0}\Phi(s) &= \infty. \end{cases}$$

Proof. Step 1 : Upper estimate. Since f satisfies (2.1.12), by [6] for any $R > 0$, there exists a solution w_R to the problem

$$\begin{cases} -\Delta w_R + f(w_R) &= 0 \qquad \text{in } B_R, \\ \lim\limits_{|x|\to R} w_R(x) &= \infty, \end{cases} \tag{2.2.9}$$

and w_R is nonnegative since $f(0) = 0$. Notice also that $R \mapsto w_R$ is decreasing, since f is nondecreasing ; moreover $\lim_{R\to\infty} w_R = 0$, since $f(0) = 0$ and f is positive on $(0,\infty)$. Let $x_0 \neq 0$ arbitrary in $\mathbb{R}^N$. Set $\mathbb{E} = \{\vec{e} : |\vec{e}| = 1\}$ and take $\vec{e} \in \mathbb{E}$. Put $x_{\vec{e}} = |x_0|\,\vec{e}$ and for $n > |x_0|$ put $a_n = n\vec{e}$. Denote by $\mathbb{H}_{\vec{e}}$ the open half-space generated by $\vec{e}$ and its orthogonal hyperplane at the origin, then $x_{\vec{e}}, a_n \in \mathbb{H}_{\vec{e}}$. Take R such that $n - |x_0| < R < n$. We set $W_{\vec{e},n,R}(x) = w_R(x - a_n)$, then $W_{\vec{e},n,R}$ is a solution of (2.1.1) in $B_R(a_n)$ and blows-up on the boundary $\lim\limits_{|x-a_n|\to R} W_{\vec{e},n,R}(x) = \infty$. By the maximum principle,

$$u_{k\delta_0}(x,t) \leq W_{\vec{e},n,R}(x) \quad \forall (x,t) \in B_R(a_n) \times (0,\infty). \tag{2.2.10}$$

The sequence $\{W_{\vec{e},n,R}\}$ is decreasing with respect to R and is bounded from below by $u_{k\delta_0}$, then there exists $W_{\vec{e},n} := \lim\limits_{R\to n} W_{\vec{e},n,R}$ satisfying

$$u_{k\delta_0}(x,t) \leq W_{\vec{e},n}(x) \quad \forall (x,t) \in B_n(a_n) \times (0,\infty). \tag{2.2.11}$$

The sequence $\{W_{\vec{e},n}\}$ is also decreasing with respect to n and is bounded from below by $u_{k\delta_0}$, then there exists $W_{\vec{e},\infty} := \lim\limits_{n\to\infty} W_{\vec{e},n}$. Letting $n \to \infty$ in (2.2.11) yields to

$$u_{k\delta_0}(x,t) \leq W_{\vec{e},\infty}(x) \quad \forall (x,t) \in \mathbb{H}_{\vec{e}} \times (0,\infty). \tag{2.2.12}$$

In particular,

$$u_{k\delta_0}(x_{\vec{e}},t) \leq W_{\vec{e},\infty}(x_{\vec{e}}). \tag{2.2.13}$$

Since $u_{k\delta_0}$ is radial, it follows that

$$u_{k\delta_0}(x_0,t) = u_{k\delta_0}(x_{\vec{e}},t) \leq W_{\vec{e},\infty}(x_{\vec{e}}).$$

For any $r > 0$, $n > r$, $n - r < R < n$ and $\vec{e}$, $\vec{e'} \in \mathbb{E}$, since w_R is radial,

$$w_R(r\vec{e} - n\vec{e}) = w_R(r\vec{e'} - n\vec{e'}).$$

Letting successively $R \to n$, $n \to \infty$ yields to

$$W_{\vec{e},\infty}(r\vec{e}) = W_{\vec{e'},\infty}(r\vec{e'}).$$

Define $\tilde{\Phi}(r) := W_{\vec{e},\infty}(r\vec{e})$, $\forall r \in (0,\infty)$ then it satisfies

$$\left\{\begin{array}{rll} -\tilde{\Phi}'' - \dfrac{N-1}{r}\tilde{\Phi}' + f(\tilde{\Phi}) & = 0 & \text{in } (0,\infty) \\ \lim\limits_{r\to 0}\tilde{\Phi}(r) & = \infty, & \end{array}\right. \tag{2.2.14}$$

and

$$u_{k\delta_0}(x,t) \le \tilde{\Phi}(|x|) \qquad \forall (x,t) \in Q_\infty. \tag{2.2.15}$$

Step 2 : End of the proof. We claim that

$$\tilde{\Phi}(r) \le \Phi(r) \qquad \forall r \in (0,\infty). \tag{2.2.16}$$

For any $\epsilon > 0$, we set $\Phi_\epsilon(r) = \Phi(r-\epsilon)$, $r > \epsilon$ then Φ_ϵ is a solution of

$$-\Phi_\epsilon'' + f(\Phi_\epsilon) = 0 \qquad \text{in } (\epsilon,\infty) \tag{2.2.17}$$

verifying $\lim\limits_{r\to\epsilon}\Phi_\epsilon(r) = \infty$. Since $\Phi_\epsilon' \le 0$, Φ_ϵ is a supersolution of the equation in (2.2.14) in (ϵ,∞), which dominates $\tilde{\Phi}$ at $r = \epsilon$. By the maximum principle, $\tilde{\Phi} \le \Phi_\epsilon$ in (ϵ,∞). Letting $\epsilon \to 0$ yields (2.2.16). Combining (2.2.15) and (2.2.16) leads to the conclusion. □

Remark. Combining Proposition 2.2.2 and Proposition 2.2.3 yields to

$$u_{k\delta_0}(x,t) \le \min\{\phi_\infty(t), \Phi(|x|)\} \qquad \forall (x,t) \in Q_\infty, \forall k > 0. \tag{2.2.18}$$

Proof of Theorem 2.1.4. Since f satisfies (2.1.12) and (2.1.16), then (2.1.10) holds. The sequence $\{u_{k\delta_0}\}$ is increasing with respect to k and is bounded from above by (2.2.18) then there exists $\underline{U} := \lim\limits_{k\to\infty} u_{k\delta_0}$ satisfying

$$\underline{U}(x,t) \le \min\{\phi_\infty(t), \Phi(|x|)\} \qquad \forall (x,t) \in Q_\infty, \forall k > 0. \tag{2.2.19}$$

Moreover, $\underline{U} \in \mathcal{U}_0$ because $\underline{U}$ has the following properties :
(i) It is positive in Q_∞, belongs to $C(\overline{Q_\infty} \setminus \{(0,0)\})$ and vanishes on $\mathbb{R}^N \times \{0\} \setminus \{(0,0)\}$.

(ii) It satisfies (2.1.1) and

$$\lim_{t\to 0}\int_{B_\sigma} \underline{U}(x,t)dx = \infty, \quad \forall \sigma > 0. \tag{2.2.20}$$

In the sense of initial trace in Definition 2.4.3, $\underline{U}$ has initial trace $tr_{\mathbb{R}^N}(\underline{U}) = (\{0\}, 0)$ (here $\{0\}$ is the singular part and the Radon measure on $\mathbb{R}^N \setminus \{0\}$ is the zero measure) and the conclusion follows from Proposition 2.4.7. □

By a simple adaptation of the proof of Proposition 2.2.2 and Proposition 2.2.3 it is possible to extend (2.2.19) to any positive solution vanishing on $\mathbb{R}^N \times \{0\} \setminus \{(0,0)\}$.

Proposition 2.2.4 *Assume* (2.1.12) *and* (2.1.15) *are satisfied. Then any positive solution* $u \in C^{2,1}(Q_\infty)$ *of* (2.1.1) *satisfies*

$$u(x,t) \leq \phi_\infty(t) \qquad \forall (x,t) \in Q_\infty. \tag{2.2.21}$$

If we assume moreover that $u \in C(\overline{Q_\infty} \setminus \{(0,0)\})$ *vanishes on* $\mathbb{R}^N \times \{0\} \setminus \{(0,0)\}$*, there holds*

$$u(x,t) \leq \min\{\phi_\infty(t), \Phi(|x|)\} \qquad \forall (x,t) \in Q_\infty. \tag{2.2.22}$$

Proof. Since (2.1.15) holds, for any $R, \tau > 0$, $(x,t) \mapsto \phi_\infty(t-\tau) + w_R(x)$ is a supersolution of (2.1.1) in $B_R \times (\tau, \infty)$. This function dominates u on the parabolic boundary, thus $u \leq \phi_\infty(.-t) + w_R$ in $B_R \times (\tau,\infty)$ by the comparison principle. Since $f(r) > 0$ if $r > 0$, $\lim_{R\to\infty} w_R = 0$ in $\mathbb{R}^N$. Therefore

$$u(x,t) \leq \phi_\infty(t) = \lim_{\tau\to 0} \lim_{R\to\infty} (\phi_\infty(t-\tau) + w_R(x)) \qquad \forall (x,t) \in Q_\infty.$$

For the second estimate we notice that (2.2.10) is valid with $u_{k\delta_0}$ replaced by u (and without assumption (2.1.8) since existence is assumed). The remaining of the proof is similar to the one of Proposition 2.2.3 and we get

$$u(x,t) \leq \Phi(|x|) \qquad \forall (x,t) \in Q_\infty.$$

This implies (2.2.22). □

It is also possible to construct a maximal element of $\mathcal{U}_0$ ($\mathcal{U}_0$ is defined in Theorem 2.1.4). For $\ell > 0$ and $\epsilon > 0$, let $u := U_{\epsilon,\ell}$ be the solution of

$$\begin{cases} \partial_t u - \Delta u + f(u) & = 0 \quad \text{in } Q_\infty \\ u(x,0) & = \ell\chi_{B_\epsilon} \quad \text{in } \mathbb{R}^N. \end{cases}$$

Proposition 2.2.5 *For any* $\tau > 0$ *and* $\epsilon > 0$*, there exist* $\ell > 0$ *and* $m(\tau,\epsilon) > 0$ *such that any positive solution* u *of* (2.1.1) *which verifies (i) in the proof of Theorem 2.1.4 satisfies*

$$u(x,t) \leq U_{\epsilon,\ell}(x,t-\tau) + m(\tau,\epsilon) \qquad \forall (x,t) \in Q_\infty,\ t \geq \tau. \tag{2.2.23}$$

Furthermore

$$\lim_{\tau\to 0} m(\tau,\epsilon) = 0 \qquad \forall \epsilon > 0. \tag{2.2.24}$$

Finally

$$\overline{U}(x,t) = \lim_{\tau\to 0} \lim_{\epsilon\to 0} \lim_{\ell\to\infty} (U_{\epsilon,\ell}(x,t-\tau) + m(\tau,\epsilon)) \tag{2.2.25}$$

is the maximal element of $\mathcal{U}_0$.

Proof. We set $\ell = \phi_\infty(\tau)$, then $u(x,\tau) \leq \ell$ for any $x \in \mathbb{R}^N$. Let $W := W_{\epsilon/2}$ be the solution of the following Cauchy-Dirichlet problem

$$\begin{cases} \partial_t W - \Delta W + f(W) & = 0 \quad \text{in } B^c_{\epsilon/2} \times (0,\infty) \\ W(x,0) & = 0 \quad \text{in } B^c_{\epsilon/2} \\ W(x,t) & = \phi_\infty(t) \quad \text{on } \partial B^c_{\epsilon/2} \times (0,\infty) \end{cases} \tag{2.2.26}$$

and put $m(\tau,\epsilon) := max\{W_{\epsilon/2}(x,\delta) : |x| > \epsilon, 0 < \delta \le \tau\}$. It is clear to see that

$$\lim_{\tau\to 0} m(\tau,\epsilon) = W_{\epsilon/2}(x,0) = 0. \tag{2.2.27}$$

From the fact that $u(x,0) = 0$ in $B^c_{\epsilon/2}$, $u(x,t) \le \phi_\infty(t)$ in $\partial B^c_{\epsilon/2}\times(0,\infty)$ and the maximum principle, it follows that $u(x,t) \le W_{\epsilon/2}(x,t)$ in $B^c_{\epsilon/2} \times (0,\infty)$.

Next, we compare $U_{\epsilon,\ell}(.,.-\tau)+m(\tau,\epsilon)$ with u in $\mathbb{R}^N \times (\tau,\infty)$. The function $U_{\epsilon,\ell}(.,.-\tau)+m(\tau,\epsilon)$ is a supersolution of (2.1.1) in $\mathbb{R}^N \times (\tau,\infty)$. If $x \in B_\epsilon$, $U_{\epsilon,\ell}(x,0) = \ell \ge u(x,\tau)$, which implies $U_{\epsilon,\ell}(x,0) + m(\tau,\epsilon) \ge u(x,\tau)$. If $x \in B^c_\epsilon$, $m(\tau,\epsilon) \ge W_{\epsilon/2}(x,\tau) \ge u(x,\tau)$, hence $U_{\epsilon,\ell}(x,0)+m(\tau,\epsilon) \ge u(x,\tau)$. So we always have $U_{\epsilon,\ell}(x,0)+m(\tau,\epsilon) \ge u(x,\tau)$ for any $x \in \mathbb{R}^N$. Applying maximum principle yields to $U_{\epsilon,\ell}(.,.-\tau)+m(\tau,\epsilon) \ge u$ in $\mathbb{R}^N \times (\tau,\infty)$. Finally, the function $\overline{U}$ defined by (2.2.25) is the maximal solution because $U_{\epsilon,\ell}(x,t-\tau) \to U_{\epsilon,\ell}(x,t)$ as $\tau \to 0$ and $U_{\epsilon,\ell} \uparrow U_{\epsilon,\infty}$ when $\ell \to \infty$ and $U_{\epsilon,\infty} \downarrow \overline{U}$ when $\epsilon \to 0$. □

2.3 About uniqueness

We prove first the existence of global radial solutions of (2.1.18) under the Keller-Osserman condition.

Proof of Proposition 2.1.5. A solution of (2.1.19) is locally given by the formula

$$w(r) = a + \int_0^r s^{1-N}\int_0^s t^{N-1}f(w)dtds \tag{2.3.1}$$

Existence follows from the Picard-Lipschitz fixed point theorem. The function is increasing and defined on a maximal interval $[0, r_a)$. By a result of Vázquez and Veron [13] $r_a = \infty$, thus the solution is global. Uniqueness on $[0,\infty)$ follows always from local uniqueness. The function $r \mapsto w(r)$ is increasing and

$$w'(r) \ge \frac{f(a)}{N}r, \qquad w(r) \ge a + \frac{f(a)}{2N}r^2$$

for all $r > 0$. □

Proposition 2.3.1 *Assume f is locally Lipschitz continuous on $\mathbb{R}_+$ and* (2.1.12) *does not hold. For any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$w_a(|x|) \le u_0(x) \le w_b(|x|) \qquad \forall x \in \mathbb{R}^N \tag{2.3.2}$$

for some $0 < a < b$, there exists a positive solution $\overline{u} \in C(\overline{Q_\infty}) \cap C^{2,1}(Q_\infty)$ of (2.1.1) *in Q_∞ and satisfying $\overline{u}(.,0) = u_0$ in $\mathbb{R}^N$. Furthermore*

$$w_a(|x|) \le \overline{u}(x,t) \le w_b(|x|) \qquad \forall (x,t) \in Q_\infty. \tag{2.3.3}$$

Proof. Clearly w_a and w_b are ordered solutions of (2.1.1). We denote by u_n the solution of the initial-boundary problem

$$\begin{cases} \partial_t u_n - \Delta u_n + f(u_n) &= 0 & \text{in } Q_n = B_n \times (0,\infty) \\ u_n(x,t) &= (w_a(|x|) + w_b(|x|))/2 & \text{on } \partial B_n \times (0,\infty) \\ u_n(x,0) &= u_0(x) & \text{in } B_n. \end{cases} \tag{2.3.4}$$

By the maximum principle, u_n satisfies (2.3.3) in Q_n. Using locally parabolic equations regularity theory, we derive that the set of functions $\{u_n\}$ is eventually equicontinuous on any compact subset of $\overline{Q_\infty}$. Using a diagonal sequence, we conclude that there exists a subsequence $\{u_{n_k}\}$ which converges locally uniformly in $\overline{Q_\infty}$ to some weak solution $\overline{u} \in C(\overline{Q_\infty})$ which satisfies $\overline{u}(.,0) = u_0$ in $\mathbb{R}^N$. By standard method, $\overline{u}$ is a strong solution (at least $C^{2,1}(Q_\infty)$). □

Proposition 2.3.2 *Assume f is locally Lipschitz continuous on $\mathbb{R}_+$ and* (2.1.10) *holds but* (2.1.12) *is not satisfied. Then for any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$0 \le u_0(x) \le w_b(|x|) \qquad \forall x \in \mathbb{R}^N \tag{2.3.5}$$

for some $b > 0$, there exists a positive solution $\underline{u} \in C(\overline{Q_\infty})$ of (2.1.1) *in Q_∞ satisfying $\underline{u}(.,0) = u_0$ in $\mathbb{R}^N$ and*

$$\underline{u}(x,t) \le \min\{\phi_\infty(t), w_b(|x|)\} \qquad \forall (x,t) \in Q_\infty. \tag{2.3.6}$$

Proof. For any $R > 0$, let u_R be the solution of

$$\begin{cases} \partial_t u_R - \Delta u_R + f(u_R) = 0 & \text{in } Q_\infty \\ \qquad u_R(x,0) = u_0(x)\chi_{B_R}(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{2.3.7}$$

The solution which is constructed is dominated by the solution of the heat equation with the same initial data. Thus

$$u_R(x,t) \le (4\pi t)^{-\frac{N}{2}} \int_{B_R} e^{-\frac{|x-y|^2}{4t}} u_0(x)dy \qquad \forall (x,t) \in Q_\infty. \tag{2.3.8}$$

and $\lim_{|x|\to\infty} u_R(x,t) = 0$ uniformly with respect to t. The functions ϕ_∞ and w_b are solutions of (2.1.1) in Q_∞, which dominate u_R at $t = 0$. By the maximum principle,

$$\min\{\phi_\infty(t), w_b(|x|)\} \ge u_R(x,t) \quad \forall (x,t) \in Q_\infty. \tag{2.3.9}$$

The fact that the mapping $R \mapsto u_R$ is increasing and (2.3.9) imply that there exists $\underline{u} := \lim_{R\to\infty} u_R$ which satisfies $\underline{u}(.,0) = u_0$ in $\mathbb{R}^N$. Letting $R \to \infty$ in (2.3.9) yields (2.3.6). □

Proof of Theorem 2.1.6. Combining Proposition 2.3.1 and Proposition 2.3.2 we see that there exists two solutions $\underline{u}$ and $\overline{u}$ with the same initial data u_0 which are ordered and different since $\lim_{|x|\to\infty} \overline{u}(x,t) = \infty$ and $\lim_{|x|\to\infty} \underline{u}(x,t) \le \phi_\infty(t) < \infty$ for all $t > 0$. □

Proof of Theorem 2.1.7. By (2.1.15), there always holds

$$(f(a) - f(b))sign(a-b) \ge f(|a-b|) \qquad \forall a, b > 0 \tag{2.3.10}$$

where

$$sign(z) = \begin{cases} 1 & \text{if } z > 0, \\ -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0. \end{cases}$$

By Kato's inequality,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| \leq [\partial_t(u - \tilde{u}) - \Delta(u - \tilde{u})]sign(u - \tilde{u}),$$

therefore by step 1,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| + f(|u - \tilde{u}|) \leq 0. \tag{2.3.11}$$

Let $\epsilon > 0$. There exists $R_\epsilon > 0$ such that for any $R \geq R_\epsilon$,

$$0 \leq |u - \tilde{u}|(x,t) \leq w_\epsilon(|x|) \quad \forall (x,t) \in B_R^c \times [0,1]. \tag{2.3.12}$$

Since w_ϵ is a positive solution of (2.1.1) which dominates $|u - \tilde{u}|$ on $\partial B_R \times [0,1]$ and at $t = 0$, it follows that $|u - \tilde{u}| \leq w_\epsilon$ in $B_R \times [0,1]$. Letting $R \to \infty$ yields to $|u - \tilde{u}| \leq w_\epsilon$ in $\mathbb{R}^N \times [0,1]$. Letting $\epsilon \to 0$ and since $\lim_{\epsilon \to 0} w_\epsilon(|x|) = 0$ for any $x \in \mathbb{R}^N$, we derive $|u - \tilde{u}| = 0$, thus $u = \tilde{u}$ in $\mathbb{R}^N \times [0,1]$. Iterating yields that equality holds in Q_∞. □

Remark. If we replace the condition (2.1.15) by the assumption that $\frac{f(s)}{s}$ is increasing in $(0, \infty)$, the conclusion of Theorem 2.1.7 remains valid.

Proof of Theorem 2.1.8.

Proof of statement (i). The solution $\underline{u}$ which is constructed in Proposition 2.3.2 is a minimal solution of (2.1.1) in Q_∞ with the initial value u_0. Indeed, if $u \in C^{2,1}(Q_\infty)$ is a nonnegative solution of (2.1.1) in Q_∞ which satisfies $u(.,0) = u_0$ in $\mathbb{R}^N$ then, by maximum principle, $u_R \leq u$ in Q_∞ where u_R is the solution of (2.3.7). Letting $R \to \infty$ yields $\underline{u} \leq u$ in Q_∞. Next we construct the maximal solution. We recall that w_R is the solution of (2.2.9). Since f satisfies (2.1.15), $w_R + u_R$ is a supersolution in $B_R \times (0,\infty)$. If $u \in C(\overline{Q_\infty})$ is a solution (2.1.1) in Q_∞ with initial data u_0, it is dominated by $w_R + u_R$ on $\partial B_R \times (0, \infty)$. Thus $u \leq w_R + u_R$ in $B_R \times (0,\infty)$. Since

$$u_R \leq u \leq w_R + u_R,$$

$w_R \to 0$ when $R \to \infty$, by Proposition 2.2.3-Step 1, and $u_R \to \underline{u}$, we derive that $u = \underline{u}$.

Proof of statement (ii). Assume that there exists at least one positive solution u of (2.1.1) satisfying (2.1.23) and $f(u) \in L^1_{loc}(\overline{Q_\infty})$, equivalently [9]

$$\int_0^\infty \int_{\mathbb{R}^N} (-u(\partial_t \eta + \Delta \eta) + f(u)\eta)\, dxdt = \int_{\mathbb{R}^N} \eta(x,0) d\mu(x) \tag{2.3.13}$$

for all $\eta \in C_c^{2,1}(\overline{Q_\infty})$. We first construct a minimal solution in the following way : let $n \in \mathbb{N}$ and $R > 0$ and let $v = v_{R,n}$ be the solution of

$$\begin{cases} \partial v - \Delta v + f(v) = 0 & \text{in } B_R \times (0,\infty) \\ v = 0 & \text{on } \partial B_R \times (0,\infty) \\ v(.,0) = u(.,2^{-n}) & \text{in } B_R. \end{cases} \tag{2.3.14}$$

By the maximum principle, $v_{R,n}(.,t) \leq u(.,t + 2^{-n})$. Furthermore,

$$v_{R,n}(x, 2^{-n}) \leq u(., 2^{-n+1}) = v_{R,n}(x,0),$$

therefore,

$$v_{R,n}(x,t+2^{-n}) \leq v_{R,n-1}(x,t) \qquad \text{in } B_R \times (0,\infty). \tag{2.3.15}$$

Using the formulation (2.3.13) with $v_{R,\epsilon}$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} (-v_{R,n}(\partial_t \eta + \Delta\eta) + f(v_{R,n})\eta)\, dxdt = \int_{\mathbb{R}^N} \eta(x,0)u(x,2^{-n})dx, \tag{2.3.16}$$

for any $\eta \in C_c^{2,1}(\overline{Q_\infty^{B_R}})$. The right-hand side of (2.3.16) converges to $\int_{\mathbb{R}^N} \eta(x,0)d\mu(x)$. Concerning the left-hand side, there holds $f(v_{R,n}(x,t)) \leq f(u(x,t+2^{-n}))$. Since $f(u) \in L^1_{loc}(\overline{Q_\infty})$, $f(v_{R,n})$ is bounded in $L^1_{loc}(\overline{Q_\infty^{B_R}})$. By the L^1 regularity theory for parabolic equations (see [8] and the references therein), the set of functions $\{v_{R,n}\}$ is locally compact in $L^1_{loc}(Q_\infty)$ and there exists a subsequence $\{n_k\}$ and a function $\underline{u}_R$ such that $v_{R,n_k} \to \underline{u}_R$, almost everywhere in $Q_\infty^{B_R}$, and $\underline{u}_R \leq u$. Noticing that the sets of functions $\{f(u(.,.+2^{-n}))\}$ and $\{u(.,.+2^{-n})\}$ are uniformly integrable, we obtain that the two sets $\{f(v_{R,n})\}$ and $\{v_{R,n}\}$ are also uniformly integrable in $B_R \times (0,T)$. It follows from Vitali's convergence theorem that, up to a subsequence still denoted by $\{n_k\}$, $v_{R,n_k} \to \underline{u}_R$ and $f(v_{R,n_k}) \to f(\underline{u}_R)$ in $L^1(B_R \times (0,T))$. Letting $n = n_k \to \infty$ in (2.3.16) we derive

$$\int_0^\infty \int_{\mathbb{R}^N} (-\underline{u}_R(\partial_t \eta + \Delta\eta) + f(\underline{u}_R)\eta)\, dxdt = \int_{\mathbb{R}^N} \eta(x,0)d\mu(x). \tag{2.3.17}$$

This means that $\underline{u}_R$ satisfies $\underline{u}_R \leq u$ and

$$\begin{cases} \partial \underline{u}_R - \Delta \underline{u}_R + f(\underline{u}_R) = 0 & \text{in } B_R \times (0,\infty) \\ \underline{u}_R = 0 & \text{on } \partial B_R \times (0,\infty) \\ \underline{u}_R(.,0) = \chi_{B_R}\mu & \text{in } B_R. \end{cases} \tag{2.3.18}$$

If $\tilde{u}$ is any other nonnegative solution of (2.1.1) in Q_∞ with initial data μ, the sequence of solutions $\tilde{v}_{R,n}$ of (2.3.14) with initial data $\tilde{u}(.,2^{-n})$ converges, up to a subsequence, to some $\underline{\tilde{u}}_R$ which satisfies $\underline{\tilde{u}}_R \leq \tilde{u}$ and is solution of problem (2.3.18). We know from [7], [8] that this problem admits at most one solution. Therefore $\underline{\tilde{u}}_R = \underline{u}_R$, which implies that $\underline{u}_R \leq \tilde{u}$ in $Q_\infty^{B_R}$. Furthermore, in the above construction, we have only used the fact that $\tilde{u}$ is defined in a domain larger than $Q_\infty^{B_R}$ and is nonnegative. Consequently, the same comparison applies if we compare $\underline{u}_R$ and $\underline{u}_{R'}$ for $R' > R$ and we obtain

$$\underline{u}_R \leq \underline{u}_{R'} \qquad \text{in } Q_\infty^{B_R}.$$

Put $\underline{u} = \lim_{R\to\infty} \underline{u}_R$. Using the monotone convergence theorem and a test function $\eta \in C_c^{2,1}(\overline{Q_\infty})$ with compact support in $Q_\infty^{B_R}$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} (-\underline{u}(\partial_t \eta + \Delta\eta) + f(\underline{u})\eta)\, dxdt = \int_{\mathbb{R}^N} \eta(x,0)d\mu(x). \tag{2.3.19}$$

from (2.3.17). Thus $\underline{u}$ satisfies (2.1.23) and $f(\underline{u}) \in L^1_{loc}(\overline{Q_\infty})$. By construction $\underline{u}$ is smaller than any other nonnegative solution.

As in the proof of statement (i), we see that, for any $n \in \mathbb{N}^*$, there holds $u \leq w_R + v_{R,n}$ in $Q_\infty^{B_R}$. Consequently $u \leq w_R + \underline{u}_R$ and letting $R \to \infty$, $u \leq \underline{u}$. Thus $u = \underline{u}$. □

2.4 Initial trace

2.4.1 The regular part of the initial trace

In this section we only assume that f is a continuous nonnegative function defined on $\mathbb{R}_+$ and that u is a $C^{2,1}$ positive solution of (2.1.1) in Q_T.

Lemma 2.4.1 *Assume G is a bounded C^2 domain in $\mathbb{R}^N$, $Q_T^{\overline{G}} := \overline{G} \times (0,T]$ and let $u \in C^{2,1}(Q_T^{\overline{G}})$ be a positive solution of (2.1.1) in Q_T^G such that $f(u) \in L^1(Q_T^G)$. Then $u \in L^\infty(0,T;L^1(G'))$ for any domain $G' \subset \overline{G'} \subset G$ and there exists a positive Radon measure μ_G on G such that*

$$\lim_{t\to 0}\int_G u(x,t)\zeta(x)dx = \int_G \zeta(x)d\mu_G(x) \qquad \forall \zeta \in C_c(G). \tag{2.4.1}$$

Proof. Let $\phi := \phi_G$ be the first eigenfunction of $-\Delta$ in $W_0^{1,2}(G)$ with corresponding eigenvalue λ_G. We assume $\phi > 0$ in G. Then

$$\frac{d}{dt}\int_G u\phi dx + \lambda_G\int_G u\phi dx + \int_G f(u)\phi\, dx + \int_{\partial G} u\frac{\partial\phi}{\partial\mathbf{n}}dS = 0$$

where $\mathbf{n}$ denote the outward normal unit vector to $\partial\Omega$. Since $\phi_{\mathbf{n}} < 0$, the function

$$t \mapsto e^{\lambda_G t}\int_G u(x,t)\phi(x)dx - \int_t^T\int_G e^{\lambda_G s}f(u)\phi dx\, ds$$

is increasing and

$$\int_G u(x,t)\phi(x)dx \le e^{\lambda_G(T-t)}\int_G u(x,T)\phi(x)dx + e^{-\lambda_G t}\int_t^T\int_G e^{\lambda_G s}f(u)\phi dx\, ds$$

for $0 < t \le T$. Thus $u \in L^\infty(0,T;L^1(G'))$ for any strict domain G' of G. If $\zeta \in C_c^2(G)$, there holds

$$\frac{d}{dt}\left(\int_G u(x,t)\zeta(x)dx - \int_t^T\int_G (f(u)\zeta - u\Delta\zeta)\, dx\, ds\right) = 0. \tag{2.4.2}$$

Consequently

$$\lim_{t\to 0}\int_G u(x,t)\zeta(x)dx = \int_G u(x,T)\zeta(x)dx + \int_0^T\int_G (f(u)\zeta - u\Delta\zeta)\, dx\, ds. \tag{2.4.3}$$

This implies that $u(.,t)$ admits a limit in $\mathcal{D}'(G)$, and this limit is a positive distribution. Therefore there exists a positive Radon measure μ_G on G satisfies (2.4.1). $\square$

Proof of Proposition 2.1.9. It is clear that $\mathcal{R}(u)$ is an open subset. If G is a strict bounded subdomain of $\mathcal{R}(u)$, i.e. $\overline{G} \subset \mathcal{R}(u)$, there exists a finite number of points z_j $(j = 1,...,k)$ with $r'_j > r_j > 0$ such that $f(u) \in L^1(Q_T^{B_{r'_j}(z_j)})$ and $\overline{G} \subset \cup_{j=1}^k B_{r_j}(z_j)$. Let $\mu_j = \mu_{B_{r_j}(z_j)}$ the measure defined in Lemma 2.4.1. If $\zeta \in C_c(G)$ there exists a partition of unity $\{\eta_j\}_{j=1}^k$ relative to the cover $\{B_{r_j}(z_j)\}_{j=1}^k$ such that $\eta_j \in C_0^\infty(G)$, $\mathrm{supp}(\eta_j) \subset B_{r_j}(z_j)$ and $\zeta = \sum_{j=1}^k \eta_j \zeta$. Since

$$\lim_{t\to 0} \int_{B_{r_j}(z_j)} u(x,t)(\eta_j\zeta)(x)dx = \int_{B_{r_j}(z_j)} (\eta_j\zeta)(x)d\mu_j(x) \qquad \forall j = 1, ..., k,$$

there exists a positive Radon measure μ on $\mathcal{R}(u)$ satisfying (2.1.24). Notice also that $u \in L^\infty(0,T; L^1(G))$ for any $G \subset \overline{G} \subset \mathcal{R}(u)$. □

The main problem is to analyse the behaviour of u on the singular set $\mathcal{S}(u)$.

2.4.2 The Keller-Osserman condition holds

If the Keller-Osserman condition holds, the existence of an initial trace of arbitrary positive solutions of (2.1.1) is based upon a dichotomy in the behaviour of those solutions near $t = 0$.

Lemma 2.4.2 *Assume u is a positive solution of* (2.1.1) *in Q_T and $z \in \mathcal{S}(u)$. Suppose that at least one of the following sets of conditions holds.*

(i) There exists an open neighborhood G of z such that $u \in L^1(Q_T^G)$.

(ii) f is nondecreasing and (2.1.12) *holds.*

Then, for every open relative neighborhood G' of z,

$$\lim_{t\to 0} \int_{G'} u(x,t)dx = \infty. \tag{2.4.4}$$

Proof. First, we assume that (i) holds and let $\zeta \in C_c^2(G)$, $\zeta \geq 0$. Since $z \in \mathcal{S}(u)$, then for every open relative neighborhood G' of z, there holds

$$\int_0^T \int_{G'} f(u)dx\,dt = \infty. \tag{2.4.5}$$

Since there exists

$$\lim_{t\to 0} \int_t^T \int_{G'} u\Delta\zeta dx\,dt = L \in \mathbb{R},$$

it follows from (2.4.3) that

$$\int_{G'} u(x,t)\zeta(x)dx = \int_t^T \int_{G'} f(u)\zeta dxds + O(1), \tag{2.4.6}$$

which implies (2.4.4).

Next we assume that (2.1.12) holds and $u \notin L^1(Q_T^G)$ for every relative neighborhood G of z. If there exists an open neighborhood $G \subset \Omega$ of z such that (2.4.4) does not hold, there exist a sequence $\{t_n\}$ decreasing to 0 and $0 \leq M < \infty$ such that

$$\sup_{t_n} \int_G u(x,t_n)dx = M. \tag{2.4.7}$$

Furthermore, we can always replace G by an open ball $B_R(z)\subset G$. Thus (2.4.7) holds with G replaced by $B_R(z)$. Put $w_{z,R}:=w_R(.-z)$ where w_R is the maximal solution of (2.2.9) and let $v:=v_n$ be the solution of

$$\left\{\begin{array}{rll}\partial_t v-\Delta v &= 0 & \text{in } B_R(z)\times(t_n,\infty)\\ v &= 0 & \text{on } \partial B_R(z)\times(t_n,\infty)\\ v(.,t_n) &= u(.,t_n) & \text{in } B_R(z).\end{array}\right. \tag{2.4.8}$$

Since $v_n\geq 0$, $f(w_{z,R}+v_n)\geq f(w_{z,R})$, and $w_{z,R}+v_n$ is a supersolution of (2.1.1) in $B_R(z)\times(t_n,T)$. It dominates u on $\partial B_R(z)\times(t_n,T)$ and at $t=t_n$, thus $u\leq w_{z,R}+v_n$ in $B_R(z)\times(t_n,T)$. We can assume that $u(.,t_n)\to\nu$ for some positive and bounded measure ν on $B_R(z)$. Therefore

$$u(x,t)\leq v(x,t)+w_{z,R}(x) \qquad \text{in } Q_T^{B_R(z)} \tag{2.4.9}$$

where v is the solution of

$$\left\{\begin{array}{rll}\partial_t v-\Delta v &= 0 & \text{in } Q_\infty^{B_R(z)}\\ v &= 0 & \text{on } \partial B_R(z)\times(0,\infty)\\ v(.,0) &= \nu & \text{in } \mathcal{D}'(B_R(z)).\end{array}\right. \tag{2.4.10}$$

Since $v\in L^1(Q_T^{B_R(z)})$ and w_R is uniformly bounded in any ball $B_{R'}(z)$ for $0<R'<R$, we conclude that $u\in L^1(Q_T^{B_{R'}(z)})$, which is a contradiction. $\square$

Definition 2.4.3 *Assume f is nondecreasing and satisfies (2.1.12). Let $u\in C^{2,1}(Q_T)$ be a positive solution of (2.1.1) in Q_T. We say that u possesses an initial trace with regular part $\mu\in\mathfrak{M}_+(\mathcal{R}(u))$ and singular part $\mathcal{S}(u)=\mathbb{R}^N\setminus\mathcal{R}(u)$ if*

(i) For any $\zeta\in C_c(\mathcal{R}(u))$,

$$\lim_{t\to 0}\int_{\mathcal{R}(u)}u(x,t)\zeta(x)dx=\int_{\mathcal{R}(u)}\zeta(x)d\mu(x). \tag{2.4.11}$$

(ii) For any open set $G\subset\mathbb{R}^N$ such that $G\cap\mathcal{S}(u)\neq\emptyset$

$$\lim_{t\to 0}\int_G u(x,t)dx=\infty. \tag{2.4.12}$$

Proof of Theorem 2.1.11. The set $\mathcal{R}(u)$ and the measure $\mu\in\mathfrak{M}_+(\mathcal{R}(u))$ are defined by Definition 2.1.10 thanks to Proposition 2.1.9. Because (2.1.12) holds, $\mathcal{S}(u)=\Omega\setminus\mathcal{R}(u)$ inherits the property (ii) in Definition 2.4.3 because of Lemma 2.4.2 (ii). $\square$

In the case $f(s)=s\ln^\alpha(s+1)$ with $\alpha>2$, Theorem 2.1.11 can be derived by adapting the technic in [5]. We present below the proof for the sake of completeness.

Lemma 2.4.4 *The function $f(r)=r\ln^\alpha(r+1)$ with $\alpha>0$ is positive and convex and it admits a conjugate function f^* defined by $f^*(s)=\max_{r>0}(rs-f(r))=\max_{r>0}(rs-r\ln^\alpha(r+1))$ which satisfies*

$$f^*(s)<se^{s^{\frac{1}{\alpha}}} \quad \forall s>0. \tag{2.4.13}$$

Proof. The second derivative of f is given by the following formula

$$f''(r) = \alpha \frac{\ln^{\alpha-2}(r+1)}{(r+1)^2}[(r+2)\ln(r+1) + (\alpha-1)r]. \tag{2.4.14}$$

It is easy to see that $f''(r) > 0$ for every $r > 0$ and the convexity of f follows straighaway. The unique maximum of $r \mapsto rs - f(r)$ is achieved for

$$s = f'(r_s) = \ln^\alpha(r_s+1) + \alpha \frac{r_s}{r_s+1}\ln^{\alpha-1}(r_s+1)$$

and satisfies $\ln^\alpha(r_s+1) \le s$. On the other hand, there exists $\theta_\alpha \ge 1$ such that, for any $r \ge 0$,

$$\ln^\alpha(r+1) + \alpha \frac{r}{r+1}\ln^{\alpha-1}(r+1) \le (\theta_\alpha + \ln(r+1))^\alpha.$$

Therefore

$$\ln^\alpha(r_s+1) \le s \le (\theta_\alpha + \ln(r_s+1))^\alpha \Longrightarrow (e^{s^{\frac{1}{\alpha}}-\theta_\alpha} - 1)_+ \le r_s \le (e^{s^{\frac{1}{\alpha}}} - 1)$$

and then

$$s(e^{s^{\frac{1}{\alpha}}-\theta_\alpha} - 1)_+(1 - \theta_\alpha s^{\frac{-1}{\alpha}})_+^\alpha \le f(r_s) \le s(e^{s^{\frac{1}{\alpha}}} - 1).$$

Finally, for every $s > 0$ and every $r > 0$,

$$rs - f(r) \le s(e^{s^{\frac{1}{\alpha}}} - 1) - s(e^{s^{\frac{1}{\alpha}}-\theta_\alpha} - 1)_+(1 - \theta_\alpha s^{\frac{-1}{\alpha}})_+^\alpha \le se^{s^{\frac{1}{\alpha}}}$$

which give the desired estimate for $f^*(s)$. □

As a consequence, it is possible to define the initial trace in the case $\alpha > 2$.

Theorem 2.4.5 *Assume $f(r) = r\ln^\alpha(r+1)$ avec $\alpha > 2$. If $u \in C^{2,1}(Q_T)$ is a positive solution of* (2.1.2) *then for every $y \in \Omega$, the following alternative holds. Either (i) for every neighborhood G of y in Ω,*

$$\lim_{t\to 0} \int_G u(x,t)dx = \infty, \tag{2.4.15}$$

or (ii) there exists an open neighborhood G of y in Ω and a positive linear functional $\mathcal{L}_G$ on $C_c^\infty(G)$ such that

$$\lim_{t\to 0} \int_G u(x,t)\zeta dx = \mathcal{L}_G(\zeta) \quad \forall \zeta \in C_c^\infty(G). \tag{2.4.16}$$

A criterion for this dichotomy result is the finiteness of the following integral

$$I_\gamma = \int_0^1 \int_G u\ln^\alpha(u+1)e^{-\phi^{-\gamma}}dxdt \tag{2.4.17}$$

where $\gamma > max\{1, \frac{2}{\alpha-2}\}$ and $\phi := \phi_G$ is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(G)$ normalized so that $\max_G \phi = 1$ and $\lambda_G > 0$ the corresponding eigenvalue.

Proof. *Case 1 :* $I_\gamma = \infty$. For $\gamma > 1$, set $\psi(x) = e^{-\phi^{-\gamma}(x)}$. Multiplying (2.1.2) by $\psi(x)$ and integrating over G we have

$$\int_G \partial_t u\, \psi dx - \int_G \Delta u\, \psi dx + \int_G \psi\, f(u) dx = 0.$$

Since $\phi(x) = 0$ on ∂G, it follows that

$$\int_G \partial_t u\, \psi dx - \int_G u \Delta \psi dx + \int_G \psi f(u) dx = 0$$

By some computations, we obtain that

$$\Delta \psi = \gamma^2 \psi \phi^{-2(\gamma+1)} |\nabla \phi|^2 - \gamma \psi [(\gamma+1)\phi^{-(\gamma+2)} |\nabla \phi|^2 + \lambda_G \phi^{-\gamma}],$$

which implies

$$\Delta \psi \leq \gamma^2 \psi \phi^{-2(\gamma+1)} |\nabla \phi|^2.$$

Therefore

$$-\frac{d}{dt} \int_G u\, \psi dx + \int_G u \gamma^2 \psi \phi^{-2(\gamma+1)} |\nabla \phi|^2 dx \geq \int_G \psi f(u) dx. \tag{2.4.18}$$

Set $\ell = max\{1, \|\nabla \phi\|_\infty^2\}$. By convexity,

$$f\left(\frac{1}{2\ell\gamma^2} u\right) \leq \frac{1}{2\ell\gamma^2} f(u).$$

By applying the above inequality with $r = \frac{1}{2\ell\gamma^2} u$ and $s = 2\ell\gamma^2 \phi^{-2(\gamma+1)}$ and and using the definition of f^*, we get

$$\begin{aligned} \textstyle\int_G u \gamma^2 \psi \phi^{-2(\gamma+1)} |\nabla \phi|^2 dx &\leq \ell \gamma^2 \textstyle\int_G \psi (f(r) + f^*(s)) dx \\ &\leq \frac{1}{2} \textstyle\int_G \psi f(u) dx + \ell \gamma^2 \textstyle\int_G \psi f^*(2\ell\gamma^2 \phi^{-2(\gamma+1)}) dx. \end{aligned} \tag{2.4.19}$$

The previous two inequalities yield to

$$-\frac{d}{dt} \int_G u \psi dx \geq \frac{1}{2} \int_G \psi f(u) dx - \ell \gamma^2 \int_G \psi f^*(2\ell\gamma^2 \phi^{-2(\gamma+1)}) dx. \tag{2.4.20}$$

By Lemma 2.4.4, the following estimate holds

$$f^*(2\ell\gamma^2 \phi^{-2(\gamma+1)}) \leq 2\ell\gamma^2 \phi^{-2(\gamma+1)} e^{(2\ell\gamma^2)^{\frac{1}{\alpha}} \phi^{\frac{-2(\gamma+1)}{\alpha}}}.$$

Since $\gamma > max\{1, \frac{2}{\alpha-2}\}$,

$$\int_G e^{-\phi^{-\gamma}(x)} \phi^{-2(\gamma+1)} e^{(2\ell\gamma^2)^{\frac{1}{\alpha}} \phi^{\frac{-2(\gamma+1)}{\alpha}}} dx < \infty.$$

Moreover $0 < \psi \leq e^{-1}$. Hence the second term on the right hand side of (2.4.20) take a finite value denoted by A. Then we obtain

$$-\frac{d}{dt} \int_G u \psi dx \geq \frac{1}{2} \int_G \psi f(u) dx - A$$

Since $I_\gamma = \infty$ then $\int_0^1 \int_G \psi f(u) dxdt = \infty$, which implies $\lim_{t\to 0} \int_G u\psi dx = \infty$, and hence $\lim_{t\to 0} \int_G u dx = \infty$.

Case 2 : $I_\gamma < \infty$. Let $K \subset\subset G$ and $\zeta \in C_c^\infty(K)$. By Hopf's lemma, there exists a positive constant c such that

$$\phi(x) \geq c\, d_{\partial G}(x)$$

where $d_{\partial G}(x)$ denotes the distance from any point $x \in G$ to ∂G. It follows that

$$\min_K \phi \geq dist(K, \partial G) =: d_K > 0.$$

Therefore

$$e^{-d_K^{-\gamma}} \int_0^1 \int_K f(u) dxdt \leq \int_0^1 \int_K \psi f(u) dxdt < \infty,$$

which implies

$$\int_0^1 \int_K f(u) dxdt < \infty.$$

Multiplying (2.1.2) by ζ and take integrating over K twice on $[t, 1]$, we obtain that

$$\int_K u(x,1)\zeta(x)dx - \int_K u(x,t)\zeta(x)dx = \int_t^1 \int_K (u\Delta\zeta - u\ln^\alpha(u+1)\zeta) dxd\tau.$$

Because the right hand side is bounded as $t \to 0$, there exists a positive linear functional $\mathcal{L}_G$ on $C_c^\infty(G)$ satisfying (2.4.16). □

If Ω is a bounded domain with a C^2 boundary and $\mu \in \mathfrak{M}^b(\Omega)$, we denote by u_μ the solution of

$$\left\{ \begin{aligned} \partial_t u - \Delta u + f(u) &= 0 \quad \text{in } Q_\infty^\Omega \\ u &= 0 \quad \text{on } \partial\Omega \times (0, \infty) \\ u(.,0) &= \mu \quad \text{in } \mathcal{D}'(\Omega). \end{aligned} \right. \tag{2.4.21}$$

We recall the following stability result proved in [8, Th 1.1].

Lemma 2.4.6 *Let Ω be a bounded domain with a C^2 boundary. Assume f is nondecreasing and satisfies* (2.1.8)*. Then for any $\mu \in \mathfrak{M}^b(\Omega)$ problem* (2.4.21) *admits a unique solution u_μ. Moreover, if $\{\mu_n\} \subset \mathfrak{M}^b(\Omega)$ converges weakly to $\mu \in \mathfrak{M}^b(\Omega)$ then $u_{\mu_n} \to u_\mu$ locally uniformly in $\overline{\Omega} \times (0, \infty)$ and in $L^1(Q_T^\Omega)$, and $f(u_{\mu_n}) \to f(u_\mu)$ in $L^1(Q_T^\Omega)$, for every $T > 0$.*

Remark. The result remains true if Ω is unbounded, with a C^2 compact (possibly empty) boundary and the μ_n have their support in a fixed compact set. In such a case $u_{\mu_n}(x,t) \to 0$ when $|x| \to \infty$, uniformly with respect to n and t since

$$|u_{\mu_n}(x,t)| \leq (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} d\,|\mu_n|\,(y) \quad \forall (x,t) \in Q_\infty. \tag{2.4.22}$$

By Lemma 2.4.6 and the remark hereafter, for every $y \in \Omega$ and $k > 0$, there exists a unique solution $v_{y,k,\Omega} := v$ to (2.4.21) with $\mu = k\delta_y$. By comparison principle (see [8, Prop

1.2]) $v_{y,k,\Omega}$ is positive, increases as k increases and depends continuously on y. Note that if $\Omega = \mathbb{R}^N$, $v_{y,k,\mathbb{R}^N}(x,t) := v_{y,k}(x,t) = u_{k\delta_0}(|x-y|\,,t)$; furthermore, if f satisfies (2.1.12), we recall that $\underline{U} = \lim_{k\to\infty} u_{k\delta_0}$ is the minimal solution of (2.1.1) in Q_∞ with initial trace $(\{0\}, 0)$.

Proposition 2.4.7 *Assume f is nondecreasing and satisfies* (2.1.8) *and* (2.1.12). *Let $u \in C^{2,1}(Q_\infty)$ is a positive solution of* (2.1.1) *in Q_∞ with initial trace $(\mathcal{S}, \mu)$. Then for every $y \in \mathcal{S}$,*

$$\underline{U}_y(x,t) := \underline{U}(x-y,t) \leq u(x,t) \tag{2.4.23}$$

in Q_∞.

Proof. By translation we may suppose that $y = 0$. Since $0 \in \mathcal{S}(u)$, for any $\eta > 0$ small enough

$$\lim_{t\to 0} \int_{B_\eta} u(x,t)dx = \infty.$$

For $\epsilon > 0$, denote $M_{\epsilon,\eta} = \int_{B_\eta} u(x,\epsilon)dx$. For any $m > m_\eta = \inf_{\sigma>0} M_{\sigma,\eta}$ there exists $\epsilon = \epsilon(m,\eta)$ such that $m = M_{\epsilon,\eta}$ and $\lim_{\eta\to 0} \epsilon(m,\eta) = 0$. Let v_η be the solution of the problem

$$\begin{cases} \partial_t v_\eta - \Delta v_\eta + f(v_\eta) = 0 & \text{in } Q_\infty \\ v_\eta(x,0) = u(x,\epsilon)\chi_{B_\eta} & \text{in } \mathbb{R}^N \end{cases}$$

where χ_{B_η} is the characteristic function of B_η. By the maximum principle $v_\eta \leq u$ in $\mathbb{R}^N \times (\epsilon, \infty)$. By Lemma 2.4.6 and the remark after v_η converges to $v_{0,m}$ when η goes to zero. Letting m go to infinity yields (2.4.23). □

Corollary 2.4.8 *Under the assumption of Proposition 2.4.7, there exists a minimal positive solution $\underline{U}_{\mathcal{S}}$ of* (2.1.1) *in Q_∞ with initial trace $(\mathcal{S}, 0)$ in the sense that*

$$\underline{U}_{\mathcal{S}}(x,t) \leq u(x,t) \qquad \forall (x,t) \in Q_\infty, \tag{2.4.24}$$

for all positive solution $u \in C^{2,1}(Q_\infty)$ of (2.1.1) *with initial trace $(\mathcal{S}, \mu)$.*

Proof. If we set $\tilde{\underline{U}}_{\mathcal{S}} = \sup\{U_y : y \in \mathcal{S}\}$, then $\tilde{\underline{U}}_{\mathcal{S}}$ is a subsolution of (2.1.1). If u is a positive solution of (2.1.1) with initial trace $(\mathcal{S}, \mu)$, then $u \geq \tilde{\underline{U}}_{\mathcal{S}}$ by Proposition 2.4.7. Therefore u is larger than the smallest solution of (2.1.1) in Q_∞ which is above $\tilde{\underline{U}}_{\mathcal{S}}$. We denote this minimal solution by $\underline{U}_{\mathcal{S}}$. □

If $\mathcal{S}$ contains some ball B_R we have a more precise result.

Proposition 2.4.9 *Assume that f satisfies* (2.1.15). *Let u be a positive solution of* (2.1.1) *in Q_∞ with initial trace $(\mathcal{S}, \mu)$. We assume that $\mathcal{S}$ has a non-empty interior, and for $R > 0$, we denote by $int_R(\mathcal{S})$ the set of $y \in \mathcal{S}$ such that $\overline{B}_R(y) \subset \mathcal{S}$. Then for any $R' \in (0,R)$ there holds*

$$\lim_{t\to 0} \frac{u(x,t)}{\phi_\infty(t)} = 1 \tag{2.4.25}$$

uniformly for $x \in \overline{B}_{R'}(y)$ and $y \in int_R(\mathcal{S})$.

Proof. Let $y \in int_R(\mathcal{S})$ and $w(x,t) = u(x,t) + w_{y,R}$ where $w_{y,R} = w_R(. - y)$ (w_R is the solution to (2.2.9)). Then w is a supersolution of (2.1.1) in $Q_\infty^{B_R(y)}$ and $\lim_{t\to 0} w(x,t) = \infty$, uniformly with respect to $x \in B_R(y)$ by (2.4.23). Then, for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $w(x,t) \geq \phi_\infty(\epsilon)$ in $Q_{t_\epsilon}^{B_R(y)}$. Since $\phi_\infty(t+\epsilon)$ remains bounded on $\partial B_R(y) \times (0,\infty)$, it follows by the maximum principle that

$$w(x,t) \geq \phi_\infty(t+\epsilon) \qquad \forall (x,t) \in Q_\infty^{B_R(y)}.$$

Letting $\epsilon \to 0$ and using the fact that $w_{y,R}$ remains uniformly bounded in $B_{R'}(y)$ with $0 < R' < R$, we derive

$$u(x,t) \geq \phi_\infty(t) - K_{R'} \qquad \forall (x,t) \in Q_\infty^{B'_R(y)}. \tag{2.4.26}$$

where $K_{R'} = \max\{w_{y,R}(x) : x \in B_{R'}(y)\}$. Combining this estimate with (2.2.21) yields to (2.4.25). □

We next derive the following convergence lemma :

Proposition 2.4.10 *Assume that f is nondecreasing and satisfies* (2.1.8)*,* (2.1.12) *and* (2.1.15)*. Let $\{u_n\}$ be a sequence of positive solutions of* (2.1.1) *in Q_∞ with initial trace $(\mathcal{S}(u_n), \mu_n)$ such that $u_n \to u$ locally uniformly in Q_∞ and let A be an open subset of $\mathcal{R}(u_n) := \mathbb{R}^N \setminus \mathcal{S}(u_n)$. Then u is a positive solution of* (2.1.1) *in Q_∞ with initial trace denoted by $tr_{\mathbb{R}^N}(u) = (\mathcal{S}, \mu)$. Furthermore, if $\mu_n(A)$ remains uniformly bounded, then $A \subset \mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$ and $\chi_A \mu_n \to \chi_A \mu$ weakly. Conversely, if $A \subset \mathcal{R}(u)$ then $\mu_n(K)$ remains bounded independently of n, for every compact set $K \subset A$.*

Proof. The fact that u is a positive solution of (2.1.1) in Q_∞ is standard by the weak formulation of the equation. Assume now that $A \cap \mathcal{S} \neq \emptyset$. Let $z \in A \cap \mathcal{S}$ and $R > 0$ such that $\overline{B}_R(z) \subset A$. By convexity, u_n is bounded from above in $Q_\infty^{B_R(z)}$ by $v_n + w_{z,R}$, where $v_{n,z}$ satisfies

$$\left\{\begin{aligned} \partial_t v - \Delta v + f(v) &= 0 && \text{in } Q_\infty^{B_R(z)} \\ v &= 0 && \text{on } \partial B_R(z) \times (0,\infty) \\ v(.,0) &= \chi_{B_R(z)} \mu_n && \text{in } B_R(z). \end{aligned}\right. \tag{2.4.27}$$

We can assume that, up to a subsequence, $\chi_{B_R(z)} \mu_{n_k} \to \mu_z \in \mathfrak{M}_+^b(B_R(z))$ weakly, thus $v_{n_k,z} \to v_z$ where v_z is the solution of

$$\left\{\begin{aligned} \partial_t v - \Delta v + f(v) &= 0 && \text{in } Q_\infty^{B_R(z)} \\ v &= 0 && \text{on } \partial B_R(z) \times (0,\infty) \\ v(.,0) &= \mu_z && \text{in } B_R(z). \end{aligned}\right. \tag{2.4.28}$$

Therefore

$$u \leq v_z + w_{z,R} \qquad \text{in } Q_\infty^{B_R(z)}. \tag{2.4.29}$$

By Lemma 2.4.6, it implies that $u \in L^1(Q_T^{B_{R'}(z)})$ for any $0 < R' < R$. Furthermore, if (2.1.8) is satisfied, then for any positive constant k, $s \mapsto s^{N/2} f(s^{-N/2} + k) \in L^1(0,1)$, thus if v is such that $f(v) \in L^1(Q_T^{B_{R'}(z)})$, there holds $f(v+k) \in L^1(Q_T^{B_{R'}(z)})$. In particular, since $f(v_z) \in L^1(Q_T^{B_{R'}(z)})$, and if we take $k = \max\{w_{z,R}(x) : x \in B_{R'}(z)\}$, we derive that $f(u) \in L^1(Q_T^{B_{R'}(z)})$, and therefore $z \in \mathcal{R}$, which is a contradiction; thus $A \subset \mathcal{R}$. Next, there exist a subsequence $\{n_k\}$ and a bounded positive measure $\tilde{\mu}$, with support in A such that $\chi_A \mu_{n_k} \to \tilde{\mu}$ weakly and suppose $\overline{B}_R(z) \subset A$. Since $u_{n_k} \le v_{n_k,z} + k$ and $f(u_{n_k}) \le f(v_{n_k,z}+k)$ in $Q_T^{B_{R'}(z)}$ and $v_{n_k,z}+k$ and $f(v_{n_k,z}+k)$ are uniformly integrable in $Q_T^{B_{R'}(z)}$, it follows that u_{n_k} and $f(u_{n_k,z})$ inherit this property. Therefore, if $\zeta \in C_c^2(B_R(z))$ we can assume that it vanishes outside $B_{R'}(z)$. Because

$$\int_{B_R(z)} \zeta(x) d\mu_{n_k}(x) = \int_{B_R(z)} u_{n_k}(x,t)\zeta(x)dx + \int_0^t \int_{B_R(z)} (-u_{n_k}\Delta\zeta + f(u_{n_k})\zeta)\, dxds, \tag{2.4.30}$$

we derive from Vitali's convergence theorem

$$\int_{B_R(z)} \zeta(x) d\tilde{\mu}(x) = \int_{B_R(z)} u(x,t)\zeta(x)dx + \int_0^t \int_{B_R(z)} (-u\Delta\zeta + f(u)\zeta)\, dxds. \tag{2.4.31}$$

This implies that $\chi_{B_R(z)}\tilde{\mu} = \chi_{B_R(z)}\mu$ and, by a partition of unity, that $\tilde{\mu} = \chi_A \mu$.

Assume now that $K \subset \mathcal{R}$ is compact. If $\mu_n(K)$ is unbounded and up to a subsequence still denoted by $\{n\}$, there exists a point $y \in K$ such that for any neighborhood $\mathcal{O}$ of y, $\mathcal{O} \subset A$, $\mu_n(\mathcal{O}) \to \infty$ as $n \to \infty$. We can take $\mathcal{O} = B_r(y)$ and put $M_{n,r} = \mu_n(B_r(y))$. If $m \in \mathbb{N}^*$, there exists an integer $n = n(m,r)$ such that $m \le M_{n,r}$, and $\lim_{r\to 0} n(m,r) = \infty$. Let $r_0 > r$ such that $B_{r_0}(y) \subset A$, and w_r be the solution of

$$\left\{\begin{array}{rl} \partial_t w - \Delta w + f(w) = 0 & \text{in } Q_\infty^{B_{r_0}(y)} \\ w = 0 & \text{on } \partial B_\infty^{B_{r_0}(y)} \\ w(.,0) = \chi_{B_r(y)}\mu_n & \text{in } B_{r_0}(y). \end{array}\right. \tag{2.4.32}$$

By the comparison principle, $w_r \le u_n$ in $Q_\infty^{B_{r_0}(y)}$. Since $\chi_{B_r(y)}\mu_n \to m\delta_y$ as $r \to 0$ and $n \to \infty$, we derive $u_{y,m,B_{r_0}(y)} \le u$ from Lemma 2.4.6 and the remark hereafter. Since m is arbitrary, $u_{y,\infty,B_{r_0}(y)} \le u$. This implies that $y \in \mathcal{S}$, which is a contradiction. □

If A is an open subset of Ω and $\nu \in \mathfrak{M}_+(A)$, we define an extension $\underline{\nu}$ of ν to Ω by

$$\underline{\nu}(E) = \inf_{E \subseteq O} \nu(O \cap A) \tag{2.4.33}$$

for every Borel set $E \subset \Omega$ where the infimum is taken over the open subsets O; $\underline{\nu}$ is an outer regular Borel measure on Ω and $\nu = \underline{\nu}_{|A}$.

The following result which shows the existence of a minimal solution of (2.1.1) with a given initial trace in $\mathfrak{M}_+(A)$ for any open subset A in $\mathbb{R}^N$ is a straightforward adaptation of [7, Lemma 3.3].

Proposition 2.4.11 *Assume that f is nondecreasing and satisfies* (2.1.8), (2.1.12) *and* (2.1.15).

(i) Let A be an open subset of $\mathbb{R}^N$ and let $\nu \in \mathfrak{M}_+(A)$ with associated extension $\underline{\nu}$. Then there exists a positive solution of (2.1.1) *in Q_∞, denoted by $\underline{u}_\nu$, satisfying $Tr_{\mathbb{R}^N}(\underline{u}_\nu) = \underline{\nu}$ and*

such that $\underline{u}_\nu \leq v$ *for every positive solution* v *of* (2.1.1) *in* Q_∞ *such that* $tr_{\mathbb{R}^N}(v) = (\mathcal{S}, \mu)$ *and* $\chi_A \mu \geq \nu$.

(ii) Let $\Omega \subset \mathbb{R}^N$ *be a bounded domain with a* C^2 *boundary and* u_n *be the solution of problem*

$$\begin{cases} \partial_t u_n - \Delta u_n + f(u_n) &= 0 \quad \text{in } Q_T^\Omega \\ u_n &= n \quad \text{on } \partial\Omega \times (0, \infty) \\ u_n(.,0) &= n \quad \text{in } \Omega. \end{cases} \tag{2.4.34}$$

Denote $U_{\infty,\Omega} := \lim_{n\to\infty} u_n$. *Then* $U_{\infty,\Omega}$ *is the maximal solution of* (2.1.1) *in* Q_∞^Ω *in the sense that the following relation holds in* Q_T^Ω *for every positive solution* v *of* (2.1.1)

$$U_{\infty,\Omega} \geq v. \tag{2.4.35}$$

Taking $A = \mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$, we obtain the existence of a minimal positive solution of (2.1.1) with a given positive Radon measure $\mu \in \mathfrak{M}_+(\mathcal{R})$ as the regular part of the initial trace.

Corollary 2.4.12 *Let* $\mathcal{S}$ *be a closed subset of* $\mathbb{R}^N$, $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$ *and* $\mu \in \mathfrak{M}_+(\mathcal{R})$. *Then there exists a positive solution* $\underline{u}_\mu$ *of* (2.1.1) *such that* $Tr_{\mathbb{R}^N}(\underline{u}_\mu) = \underline{\mu}$ *and* $\underline{u}_\mu \leq v$ *for every positive solution* v *of* (2.1.1) *in* Q_∞ *such that* $tr_{\mathbb{R}^N}(v) = (\mathcal{S}, \mu)$.

As a counterpart of Theorem 2.1.11 we have a existence result stated in Theorem 2.1.12.

Proof of Theorem 2.1.12.

Step 1 : Construction of a minimal solution. Let $\underline{U}_\mathcal{S}$ and $\underline{u}_\mu$ the minimal solution constructed in Corollary 2.4.8 and Corollary 2.4.12. Then $\underline{\check{u}}_{\mathcal{S},\mu} := \sup\{\underline{U}_\mathcal{S}, \underline{u}_\mu\}$ is a subsolution of (2.1.1) in Q_∞ while $\underline{\hat{u}}_{\mathcal{S},\mu} := \underline{U}_\mathcal{S} + \underline{u}_\mu$ is a supersolution. Furthermore $\underline{\check{u}}_{\mathcal{S},\mu} \leq \underline{\hat{u}}_{\mathcal{S},\mu}$. Therefore the set of solutions u in Q_∞ such that $\underline{\tilde{u}}_{\mathcal{S},\mu} \leq u \leq \underline{\hat{u}}_{\mathcal{S},\mu}$ is not empty and we denote by $\underline{u}_{\mathcal{S},\mu}$ the smallest solution larger than $\underline{\check{u}}_{\mathcal{S},\mu}$; it is a solution with initial trace $(\mathcal{S}, \mu)$. If u is any other positive solution with the same initial trace, it is larger than $\underline{u}_\mathcal{S}$ and $\underline{u}_\mu$ by Corollary 2.4.8 and Corollary 2.4.12. Therefore it is larger than $\underline{\check{u}}_{\mathcal{S},\mu}$ and consequently larger than $\underline{u}_{\mathcal{S},\mu}$.

Step 2 : Construction of the maximal solution. The proof is somewhat similar to the one on [7, Th 3-4], but we give it for the sake of completeness. We denote, for $\delta > 0$,

$$\mathcal{S}^\delta := \{x \in \mathbb{R}^N : \text{dist}\,(x, \mathcal{S}) \leq \delta\} \text{ and } \mathcal{R}^\delta := \mathbb{R}^N \setminus \mathcal{S}^\delta.$$

and let μ_δ be the measure given by

$$\mu_\delta(E) = \mu(\mathcal{R}_\delta \cap E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}.$$

We denote by $u_{\mathcal{S}^\delta}$ a solution of (2.1.1) in Q_∞ with initial trace $(\mathcal{S}^\delta, 0)$: a solution is easily constructed as the limit when $R, k \to \infty$ of the solution $v = v_{k,R}$ of

$$\begin{cases} \partial_t v - \Delta v + f(v) &= 0 \qquad \text{in } Q_\infty \\ v(.,0) &= k\chi_{(\overline{B}_R \cap \mathcal{S}^\delta) \cup (\overline{B}_R \cap \overline{B}^c_{R-\delta})} \quad \text{in } \mathbb{R}^N \end{cases} \tag{2.4.36}$$

By Proposition 2.4.9, there holds, for any $0 < \delta' < \delta$ and $\epsilon > 0$,

$$\lim_{t\to 0} \frac{u_{\mathcal{S}^\delta}(x,t)}{\phi_\infty(t)} = 1 \quad \text{uniformly on } \mathcal{S}_{\delta'} \tag{2.4.37}$$

Let u_{μ_δ} be the solution of of (2.1.1) in Q_∞ with initial trace $(\emptyset, \mu_\delta)$. This solution is constructed by approximation, as the limit, when $R \to \infty$, of the solution $u = u_{\chi_{B_R}\mu_\delta}$ of

$$\left\{ \begin{array}{rl} \partial_t u - \Delta u + f(u) = 0 & \text{in } Q_\infty \\ u(.,0) = \chi_{B_R}\mu_\delta & \text{in } \mathbb{R}^N. \end{array} \right. \tag{2.4.38}$$

For $\tau > 0$, let $u_{\delta,\tau}$ be the solution of (2.1.1) in Q_∞ with initial data $m_{\delta,\tau}$ defined by

$$m_{\delta,\tau}(x) = \left\{ \begin{array}{ll} \phi_\infty(\tau) & \text{if } x \in \mathcal{S}_\delta \\ u_{\mu_\delta}(x,\tau) & \text{if } x \in \mathcal{R}_\delta. \end{array} \right.$$

Then $u(.,\tau) \leq m_{\delta,\tau}$ in $\mathcal{S}_\delta$ and $u(.,\tau) \geq m_{\delta,\tau}$ in $\mathcal{R}_\delta$ by Proposition 2.4.11. Therefore

$$\lim_{\tau\to 0} (u(.,\tau) - m_{\delta,\tau}(.))_+ = 0$$

in the weak sense of measures. Furthermore, this solution does not depend on u, but only on $\mathcal{S}_\delta$ and μ_δ. The set of functions $\{u_{\delta,\tau}\}_{\tau>0}$ is locally uniformly bounded in Q_∞. By the regularity theory for parabolic equations, there exists a subsequence $\{\tau_k\}$ and a positive solution u^*_δ of (2.1.1) in Q_∞ such that $u_{\delta,\tau_k} \to u^*_\delta$ locally uniformly in Q_∞. By Proposition 2.4.9 and Proposition 2.4.11, $tr_{\mathbb{R}^N}(u^*_\delta) = (\mathcal{S}^\delta, \mu_\delta)$. Let $\omega_{\delta,\tau}$ be the solution of (2.1.1) in Q_∞ with initial data $(u(.,\tau) - m_{\delta,\tau}(.))_+$ (it is constructed in the same way as $\underline{u}_\mu$ in Proposition 2.4.11 -(i)). By Theorem 2.1.8-(ii), $\lim_{\tau\to 0} \omega_{\delta,\tau} = 0$, locally uniformly. Since $u \leq u_{\delta,\tau} + \omega_{\delta,\tau}$ in $(\tau,\infty) \times \mathbb{R}^N$, we obtain $u \leq u^*_\delta$. If $0 < \delta' < \delta$, we can compare similarly $u_{\delta,\tau}$ with the solution $u_{\delta',\tau}$ of (2.1.1) with initial data

$$m_{\delta',\tau}(x) = \left\{ \begin{array}{ll} \phi_\infty(\tau) & \text{if } x \in \mathcal{S}'_\delta \\ u_{\mu_{\delta'}}(x,\tau) & \text{if } x \in \mathcal{R}'_\delta. \end{array} \right.$$

If $u^*_{\delta'}$ is the limit of any sequence $\{u_{\delta',\tau_{k'}}\}$, it satisfies $0 < u^*_{\delta'} \leq u^*_\delta$ and has initial trace $(\mathcal{S}^{\delta'}, \mu_{\delta'})$. By taking in particular $\delta = \delta_n = 2^{-n}$, we construct a decreasing sequence of positive solutions $\{u^*_{2^{-n}}\}$ of (2.1.1) in Q_∞, with $tr_{\mathbb{R}^N}(u^*_{2^{-n}}) = (\mathcal{S}^{2^{-n}}, \mu_{2^{-n}})$, satisfying

$$u \leq u^*_{2^{-n}} \qquad \text{in } Q_\infty.$$

Clearly the limit $\overline{u}_{\mathcal{S},\mu}$ of the sequence $\{u^*_{2^{-n}}\}$ is a positive solution of (2.1.1) in Q_∞ with initial trace $(\mathcal{S},\mu)$ and is independent of u. It is the maximal solution of the equation with this initial trace. □

Remark. When $f(r) = |r|^{q-1}r$ with $1 < q < 1+2/N$, precise expansion of $u_{\infty\delta_0}(x,t)$, when $t \to 0$ allows to prove uniqueness [7]. Even when $f(r) = r\ln^\alpha(r+1)$ with $\alpha > 2$, uniqueness is not known. However, if $\mathcal{S} = \emptyset$, uniqueness holds from Theorem 2.1.8-(ii).

2.4.3 The Keller-Osserman condition does not hold

In this section we assume that (2.1.12) does not hold but (2.1.8) is satisfied.

Lemma 2.4.13 *Assume* (2.1.8) *and* (2.1.10) *are satisfied but* (2.1.12) *is not satisfied. Moreover assume that* $\lim_{k\to\infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ *for every* $(x,t) \in Q_\infty$. *If* u *is a positive solution of* (2.1.1) *in* Q_∞ *which satisfies*

$$\limsup_{t\to 0} \int_G u(x,t)dx = \infty, \tag{2.4.39}$$

for some bounded open subset $G \subset \mathbb{R}^N$*, then* $u(x,t) \geq \phi_\infty(t)$ *for every* $(x,t) \in Q_\infty$*. This holds in particular if* $f(r) = r\ln^\alpha(r+1)$ *with* $1 < \alpha \leq 2$*.*

Proof. By assumpion, there exists a sequence $\{t_n\}$ decreasing to 0 such that

$$\lim_{n\to\infty} \int_G u(x,t_n)dx = \infty. \tag{2.4.40}$$

Since (2.4.39) holds, we can construct a decreasing sequence of open subsets $G_k \subset G$ such that $\overline{G_k} \subset G_{k-1}$, $\operatorname{diam}(G_k) = \epsilon_k \to 0$ when $k \to \infty$, and

$$\lim_{n\to\infty} \int_{G_k} u(x,t_n)dx = \infty \qquad \forall k \in \mathbb{N}. \tag{2.4.41}$$

Furthermore there exists a unique $a \in \cap_k G_k$. We set

$$\int_{G_k} u(x,t_n)dx = M_{n,k}.$$

Since $\lim_{n\to\infty} M_{n,k} = \infty$, we claim that for any $m > 0$ and any k, there exists $n = n(k) \in \mathbb{N}$ such that

$$\int_{G_k} u(x,t_{n(k)})dx \geq m. \tag{2.4.42}$$

By induction, we define $n(1)$ as the smallest integer n such that $M_{n,1} \geq m$. This is always possible. Then we define $n(2)$ as the smallest integer larger than $n(1)$ such that $M_{n,2} \geq m$. By induction, $n(k)$ is the smallest integer n larger than $n(k-1)$ such that $M_{n,k} \geq m$. Next, for any k, there exists $\ell = \ell(k)$ such that

$$\int_{G_k} \inf\{u(x,t_{n(k)});\ell\}dx = m \tag{2.4.43}$$

and we set

$$V_k(x) = \inf\{u(x,t_{n(k)});\ell\}\chi_{G_k}(x).$$

Let $v_k = v$ be the unique bounded solution of

$$\left\{\begin{array}{rl} \partial_t v - \Delta v + f(v) = 0 & \text{in } Q_\infty \\ v(.,0) = V_k & \text{in } \mathbb{R}^N. \end{array}\right. \tag{2.4.44}$$

Since $v(x,0) \leq u(x,t_{n(k)})$, we derive

$$u(x,t+t_{n(k)}) \geq v_k(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{2.4.45}$$

When $k \to \infty$, $V_k \to m\delta_a$, thus $v_k \to u_{m\delta_a}$ by Lemma 2.4.6. Therefore $u \geq u_{m\delta_a}$. Since m is arbitrary and $u_{m\delta_a} \to \phi_\infty$ when $m \to \infty$ by Theorem 2.1.3, it follows that $u \geq \phi_\infty$. □

Lemma 2.4.14 *Assume* (2.1.8) *is satisfied but neither* (2.1.10) *nor* (2.1.12) *is satisfied. Moreover assume that* $\lim_{k\to\infty} u_{k\delta_0} = \infty$ *in* Q_∞. *There exists no positive solution* u *of* (2.1.1) *in* Q_∞ *which satisfies* (2.4.39) *for some bounded open subset* $G \subset \mathbb{R}^N$. *This holds in particular if* $f(r) = r\ln^\alpha(r+1)$ *with* $0 < \alpha \leq 1$.

Proof. If we assume that such a u exists, we proceed as in the proof of the previous lemma. Since Lemma 2.4.6 holds, we derive that $u \geq u_{m\delta_a}$ for any m. Since $\lim_{m\to\infty} u_{m\delta_a}(x,t) = \infty$ for all $(x,t) \in Q_\infty$, we are led to a contradiction. □

Thanks to these results, we can characterize the initial trace of positive solutions of (2.1.1) when the Keller-Osserman condition does not hold.

Proof of Theorem 2.1.13. If there exists some open subset G of $\mathbb{R}^N$ with the property (2.4.39), then $u \geq \phi_\infty$ and the initial trace of u is the Borel measure ν_∞. Next we assume that for any bounded open subset G of $\mathbb{R}^N$ there holds

$$\limsup_{t\to 0} \int_G u(x,t)dx < \infty. \tag{2.4.46}$$

If $\mathcal{S}(u) \neq \emptyset$, there exist $z \in \mathbb{R}^N$ and a bounded open neighborhood G of z such that

$$\int_0^T \int_G f(u)dxdxt = \infty.$$

By (2.4.46), $u \in L^\infty(0,T;L^1(G)) \subset L^1(Q_T^G)$. Then, by Lemma 2.4.2, (2.4.4) holds, which contradict (2.4.46). Thus $\mathcal{S}(u) = \emptyset$ and $\mathcal{R}(u) = \mathbb{R}^N$. It follows from Proposition 2.1.9 that there exists a positive Radon measure μ such that

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N} \zeta(x)d\mu(x) \qquad \forall \zeta \in C_c(\mathbb{R}^N). \tag{2.4.47}$$

□

Because of the lack of uniqueness from Theorem 2.1.6 it is difficult to give a complete characterization of admissible initial data for solutions of (2.1.1) under the assumptions of Theorem 2.1.13. However, we have the result as in Proposition 2.1.14.

Proof of Proposition 2.1.14. We first notice that $\max\{\phi_\infty(t), w_b(|x|)\}$ is a subsolution of (2.1.1) which is dominated by the supersolution $\phi_\infty(t) + w_b(|x|)$. The construction is standard : for $\tau > 0$ we set

$$\psi(x,\tau) = \frac{1}{2}\left(\max\{\phi_\infty(t), w_b(|x|)\} + \phi_\infty(t) + w_b(|x|)\right).$$

There exists a function $u = u_\tau \in C(\overline{Q_\infty})$ solution of (2.1.1) in Q_∞ such that $u_\tau(.,0) = \psi(.,\tau)$. Furthermore

$$\max\{\phi_\infty(t+\tau), w_b(|x|)\} \leq u_\tau(x,t) \leq \phi_\infty(t+\tau) + w_b(|x|) \qquad \forall (x,t) \in Q_\infty. \tag{2.4.48}$$

By the parabolic equation regularity theory, the set $\{u_\tau\}_{\tau>0}$ is locally equicontinuous in Q_∞. Thus there exist a subsequence $\{\tau_n\}$ and $u \in C(Q_\infty)$ such that $u_{\tau_n} \to u$ on any compact subset of Q_∞. Clearly u is a weak, thus a strong solutions of (2.1.1) and it satisfies (2.1.28). Since any solution u with initial trace ν_∞ dominates ϕ_∞ by Lemma 2.4.13, it follows that ϕ_∞ is the minimal one. □

Proof of Theorem 2.1.15. As in the proof of Theorem 2.1.13 and due to Lemma 2.4.14, $\mathcal{S}(u) = \emptyset$. Therefore $\mathcal{R}(u) = \mathbb{R}^N$ and the proof follows from Proposition 2.1.9. □

Remark. Under the assumptions of Theorem 2.1.13, it is clear, from the proof of Proposition 2.3.1, that for any $0 < a < b$ and any initial data $u_0 \in C(\mathbb{R}^N)$ satisfying

$$w_a(x) \leq u_0(x) \leq w_b(x) \qquad \forall x \in \mathbb{R}^N$$

there exists a solution $u \in C(\overline{Q_\infty})$ of (2.1.1) in Q_∞ satisfying $u(.,0) = u_0$ and

$$w_a(x) \leq u(x,t) \leq w_b(x) \qquad \forall (x,t) \in Q_\infty.$$

We conjecture that for any positive measure μ on $\mathbb{R}^N$ which satisfies, for some $b > 0$,

$$\int_{B_R} d\mu(x) \leq \int_{B_R} w_b(x)dx \qquad \forall R > 0 \tag{2.4.49}$$

there exists a positive solution u of (2.1.1) in Q_∞ with initial trace μ. Another interesting open problem is to see if there exist local solutions in Q_T with an initial trace μ satisfying

$$\lim_{R\to\infty} \frac{\displaystyle\int_{B_R} d\mu(x)}{\displaystyle\int_{B_R} w_b(x)dx} = \infty \qquad \forall b > 0. \tag{2.4.50}$$

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Chapitre 3

Initial trace of positive solutions of a class of degenerate heat equation with absorption

Abstract

We study the initial value problem with unbounded nonnegative functions or measures for the equation $\partial_t u - \Delta_p u + f(u) = 0$ in $\mathbb{R}^N \times (0, \infty)$ where $p > 1$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u)$ and f is a continuous, nondecreasing nonnegative function such that $f(0) = 0 = f^{-1}(0)$. In the case $p > \frac{2N}{N+1}$, we provide a sufficient condition on f for existence and uniqueness of the solutions satisfying the initial data $k\delta_0$ and we study their limit when $k \to \infty$, according f^{-1} and $F^{-1/p}$ are integrable or not at infinity, where $F(s) = \int_0^s f(r)dr$. We also give new results dealing with non uniqueness for the initial value problem with unbounded initial data. If $p > 2$, we prove that, for a large class of nonlinearities f, any positive solution admits an initial trace in the class of positive Borel measures. As a model case we consider the case $f(u) = u^\alpha \ln^\beta(u+1)$, where $\alpha > 0$ and $\beta > 0$.

3.1 Introduction

The aim of this chapter is to study some qualitative properties of the positive solutions of

$$\partial_t u - \Delta_p u + f(u) = 0 \tag{3.1.1}$$

in $Q_\infty := \mathbb{R}^N \times (0,\infty)$ where $p > 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and f is a continuous, nondecreasing function such that $f(0) = 0 = f^{-1}(0)$. The properties we are interested in are mainly : (a) the existence of fundamental solutions i.e. solutions with $k\delta_0$ as initial data and the behaviour of these solutions when $k \to \infty$; (b) the existence of an initial trace and its properties ; (c) uniqueness and non-uniqueness results for the Cauchy problem. This type of questions have been considered in a previous chapter in the semilinear case $p = 2$. The breadcrumbs of this study lies in the existence of two types of specific solutions of (3.1.1). The first ones are the solutions $\phi := \phi_a$ of the ODE

$$\phi' + f(\phi) = 0 \tag{3.1.2}$$

defined on $[0,\infty)$ and subject to $\phi(0) = a \geq 0$; it is given by

$$\int_{\phi(t)}^{a} \frac{ds}{f(s)}. \tag{3.1.3}$$

The second ones are the solutions of the elliptic equation

$$-\Delta_p w + f(w) = 0, \tag{3.1.4}$$

defined in $\mathbb{R}^N$ or in $\mathbb{R}^N \setminus \{0\}$. It is well-known that the structure of the set of solutions of (3.1.2) depends whether the following quantity

$$J := \int_1^\infty \frac{ds}{f(s)} \tag{3.1.5}$$

is finite or infinite. If $J < \infty$ there exists a maximal solution ϕ_∞ to (3.1.2) defined on $(0,\infty)$ while no such solution exists if $J = \infty$ since $\lim_{a\to\infty} \phi_a(t) = \infty$. This maximal solution plays an important role since, by the maximum principle, it dominates any solution u of (3.1.1) which satisfies

$$\lim_{|x|\to\infty} u(x,t) = 0 \tag{3.1.6}$$

for all $t > 0$, locally uniformly on $(0,\infty)$. Concerning (3.1.4) we associate the quantity

$$K := \int_1^\infty \frac{ds}{F(s)^{1/p}}. \tag{3.1.7}$$

It is a consequence of the Vázquez's extension of the Keller-Osserman condition (see [17], [12]) that if $K < \infty$, equation (3.1.4) admits a maximal solution $W_{\mathbb{R}^N_*}$ in $\mathbb{R}^N \setminus \{0\}$. This solution is constructed as the limit, when $R \to \infty$ and $\epsilon \to 0$ of the solution $W := W_{\epsilon,R}$ of (3.1.4) in $\Gamma_{\epsilon,R} := B_R \setminus \overline{B}_\epsilon$, subject to the conditions $\lim_{|x|\downarrow\epsilon} W_{\epsilon,R}(x) = \infty$ and $\lim_{|x|\uparrow R} W_{\epsilon,R}(x) = \infty$. On the contrary, if $K = \infty$, such functions $W_{\epsilon,R}$ and $W_{\mathbb{R}^N_*}$ do not exist, a situation which will be exploited in Section 3 for proving existence of global solutions of (3.1.4) in $\mathbb{R}^N$. An additional natural growth assumption of f that will be often made is the *super-additivity*

$$f(s+s') \geq f(s) + f(s') \qquad \forall s, s' \geq 0, \tag{3.1.8}$$

which, combined with the monotonicity of f, implies a minimal linear growth at infinity

$$\liminf_{s\to\infty} \frac{f(s)}{s} > 0. \tag{3.1.9}$$

If $p \geq 2$, $K < \infty$ jointly with (3.1.8) implies $J < \infty$, but this does not hold when $1 < p < 2$. When $p > 2$, $J < \infty$ and $K < \infty$, Kamin and Vázquez proved universal estimates for solutions which vanish on $\mathbb{R}^N \times \{0\} \setminus \{(0,0)\}$ (see [11]). By a slight modification of the proof of Proposition 2.2.4 in the chapter 2, it is possible to extend their result to the case $p > 1$.

Proposition (Universal estimates) *Assume $p > 1$ and f satisfies $K < \infty$. Let $u \in C(\overline{Q_\infty} \setminus \{(0,0)\})$ be a solution of* (3.1.1) *in Q_∞, which vanishes on $\mathbb{R}^N \times \{0\} \setminus \{(0,0)\}$. Then*

$$u(x,t) \leq W_{\mathbb{R}^N_*}(x) \qquad \forall (x,t) \in Q_\infty. \tag{3.1.10}$$

If we suppose moreover $J < \infty$ and that (3.1.8) *holds, then*

$$u(x,t) \leq \min\left\{\phi_\infty(t), W_{\mathbb{R}^N_*}(x)\right\} \qquad \forall (x,t) \in Q_\infty. \tag{3.1.11}$$

When $K = \infty$, no such estimate exists since the function w_a solution of (3.1.16) is a stationnary solution of (3.1.1) with unbounded initial data.

In Section 2 we study the existence of the fundamental solutions $u_{k\delta_0}$ and their behaviour when $k \to \infty$. Kamin and Vázquez proved in [11, Lemma 2.3 and Lemma 2.4], that if $p > 2$ and

$$\int_1^\infty s^{-p-\frac{p}{N}} f(s)ds < \infty, \tag{3.1.12}$$

then for any $k > 0$, there exists a unique positive solution $u := u_{k\delta_0}$ to problem

$$\left\{\begin{aligned} \partial_t u - \Delta_p u + f(u) &= 0 && \text{in } Q_\infty \\ u(.,0) &= k\delta_0 && \text{in } \mathbb{R}^N. \end{aligned}\right. \tag{3.1.13}$$

Furthermore the mapping $k \mapsto u_{k\delta_0}$ is increasing. Their existence proof heavily relies on the fact that, if we denote by $v := v_{k\delta_0}$ the fundamental (or Barenblatt-Prattle) solution of

$$\left\{\begin{aligned} \partial_t v - \Delta_p v &= 0 && \text{in } Q_\infty \\ v(.,0) &= k\delta_0 && \text{in } \mathbb{R}^N, \end{aligned}\right. \tag{3.1.14}$$

then $v_{k\delta_0}(.,t)$ is compactly supported in some ball $B_{\delta_k(t)}$, where $\delta_k(t)$ is explicit. Since $v_{k\delta_0}$ is a natural supersolution for (3.1.13), condition (3.1.12) states that $f(v_{k\delta_0}) \in L^1_{loc}(\overline{Q_\infty})$. When $2N/(N+1) < p \leq 2$, $v_{k\delta_0}(x,t) > 0$ for all $(x,t) \in Q_\infty$. It is already proved in Proposition 2.2.1 in the chapter 2 that, when $p = 2$, condition (3.1.12) yields to $f(v_{k\delta_0}) \in L^1(Q_T)$. We prove here that this result also holds when $2N/(N+1) < p < 2$ and more precisely,

Theorem 3.1.1 *Assume* $p > \frac{2N}{N+1}$ *and* f *satisfies* (3.1.12). *Then there exists a unique positive solution* $u := u_{k\delta_0}$ *to problem* (3.1.13).

In view of this result and the *a priori* estimates (3.1.10) and (3.1.11), it is natural to study the limit of $u_{k\delta_0}$ when $k \to \infty$. We denote by $\mathcal{U}_0$ the set of positive $u \in C(\overline{Q_\infty} \setminus \{(0,0)\})$ which are solutions of (3.1.1) in Q_∞, vanishes on the set $\{(x,0) : x \neq 0\}$ and satisfies

$$\lim_{t \to 0} \int_{B_\epsilon} u(x,t)dx = \infty \quad \forall \epsilon > 0.$$

Theorem 3.1.2 *Assume* $p > \frac{2N}{N+1}$, $J < \infty$, $K < \infty$ *and the condition* (3.1.12) *holds. Then* $\underline{U} = \lim_{k \to \infty} u_{k\delta_0}$ *exists and it is the smallest element of* $\mathcal{U}_0$.

When one, at least, of the above properties on J and K fails, the situation is much more complicated and fairly well understood only in the case where f has a power-like or a logarithmic-power-like growth. We first note that

(A) If $f(s) \sim s^\alpha$ $(\alpha > 0)$, then $J < \infty$ if and only if $\alpha > 1$, while $K < \infty$ if and only if $\alpha > p - 1$. Moreover (3.1.12) holds if and only if $\alpha < p(1 + \frac{1}{N}) - 1$.

(B) If $f(s) \sim s^\alpha \ln^\beta(s+1)$ $(\alpha, \beta > 0)$, then $J < \infty$ if and only if $\alpha > 1$ and $\beta > 0$, or $\alpha = 1$ and $\beta > 1$ while $K < \infty$ if and only if $\alpha > p-1$ and $\beta > 0$, or $\alpha = p-1$ and $\beta > p$. Moreover (3.1.12) holds if and only if $\alpha < p(1 + \frac{1}{N}) - 1$ and $\beta > 0$.

Theorem 3.1.3 *Assume* $p > 2$ *and* $f(s) = s^\alpha \ln^\beta(s+1)$ *where* $\alpha \in (1, p-1)$ *and* $\beta > 0$. *Let* $u_{k\delta_0}$ *be the solution of* (3.1.13). *Then* $\lim_{k \to \infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ *for every* $(x,t) \in Q_\infty$.

When $\alpha = 1$ the following phenomenon occurs.

Theorem 3.1.4 *Assume* $p > 2$ *and* $f(s) = s \ln^\beta(s+1)$ *with* $\beta > 0$. *Let* $u_{k\delta_0}$ *be the solution of* (3.1.13). *Then*

(i) If $\beta > 1$ *then* $\lim_{k \to \infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ *for every* $(x,t) \in Q_\infty$,

(ii) If $0 < \beta \leq 1$ *then* $\lim_{k \to \infty} u_{k\delta_0}(x,t) = \infty$ *for every* $(x,t) \in Q_\infty$.

Section 3 is devoted to study non-uniqueness of solutions of (3.1.1) with unbounded initial data. The starting observation is the following global existence result for solutions of (3.1.4) :

Theorem 3.1.5 *Assume* $p > 1$, f *is locally Lipschitz continuous and* $K = \infty$. *Then for any* $a > 0$, *there exists a unique solution* $w := w_a$ *to the problem*

$$-(r^{N-1}|w_r|^{p-2}w_r)_r + r^{N-1}f(w) = 0 \tag{3.1.15}$$

defined on $[0, \infty)$ *and satisfying* $w(0) = a$, $w_r(0) = 0$. *It is given by*

$$w_a(r) = a + \int_0^r H_p\left(s^{1-N} \int_0^s \tau^{N-1} f(w_a(\tau))d\tau\right) ds \tag{3.1.16}$$

where H_p *is the inverse function of* $t \mapsto |t|^{p-2}t$.

This result extends to the general case $p > 1$ a previous theorem of Vázquez and Véron [18] obtained in the case $p = 2$. The next theorem extends to the case $p \neq 2$ the theorem 2.1.6 in chapter 2 for the case $p = 2$.

Theorem 3.1.6 *Assume $p > \frac{2N}{N+1}$, f is locally Lipschitz continuous, $J < \infty$ and $K = \infty$. For any function $u_0 \in C(Q_\infty)$ which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \tag{3.1.17}$$

for some $0 < a < b$, there exist at least two solutions $\underline{u}, \overline{u} \in C(\overline{Q_\infty})$ of (3.1.1) with initial value u_0. They satisfy respectively

$$0 \leq \underline{u}(x,t) \leq \min\{w_b(|x|), \phi_\infty(t)\} \quad \forall (x,t) \in Q_\infty,$$

thus $\lim_{t\to\infty} \underline{u}(x,t) = 0$, uniformly with respect to $x \in \mathbb{R}^N$, and

$$w_a(|x|) \leq \overline{u}(x,t) \leq w_b(|x|) \quad \forall (x,t) \in Q_\infty,$$

thus $\lim_{|x|\to\infty} \overline{u}(x,t) = \infty$, uniformly with respect to $t \geq 0$.

In section 4 we prove an existence and stability result for the initial value problem

$$\begin{cases} \partial_t u - \Delta_p u + f(u) & = 0 \quad \text{in } Q_\infty \\ u(.,0) & = \mu \quad \text{in } \mathbb{R}^N \end{cases} \tag{3.1.18}$$

where $\mu \in \mathfrak{M}^b_+(\mathbb{R}^N)$, the set of positive and bounded Radon measures in $\mathbb{R}^N$.

Theorem 3.1.7 *Assume $p > \frac{2N}{N+1}$ and f satisfies (3.1.12). Then for any $\mu \in \mathfrak{M}^b_+(\mathbb{R}^N)$ the problem (3.1.18) admits a weak solution u_μ. Moreover, if $\{\mu_n\}$ is a sequence of functions in $L^1_+(\mathbb{R}^N)$ with compact support, which converges to $\mu \in \mathfrak{M}^b_+(\mathbb{R}^N)$ in the weak sense of measures, then the corresponding solutions $\{u_{\mu_n}\}$ of (3.1.18) with initial data μ_n converge to some solution u_μ of (3.1.18), strongly in $L^1_{loc}(\overline{Q_T})$ and locally uniformly in $Q_T := \mathbb{R}^N \times (0,T)$. Furthermore $\{f(u_{\mu_n})\}$ converges strongly to $f(u_\mu)$ in $L^1_{loc}(\overline{Q_T})$.*

In Section 5, we discuss the *initial trace* of positive *weak solution* of (3.1.1). The power case $f(u) = u^q$ with $q > 0$ was investigated by Bidaut-Véron, Chasseigne and Véron in [2]. They proved the existence of an initial trace in the class of positive Borel measures according to the different values of $p-1$ and q. Accordingly they studied the corresponding Cauchy problem with a given Borel measure as initial data. However their method was strongly based upon the fact that the nonlinearity was a power, which enabled to use Hölder inequality in order to show the domination of the absorption term over the other terms. In the present paper, we combine the ideas in [2] and [15] with a stability result for the Cauchy problem and Harnack's inequality in the form of [5] to establish the following dichotomy result which is new even in the case $p = 2$.

Theorem 3.1.8 *Assume $p \geq 2$ and (3.1.12) holds. Let $u \in C(Q_T)$ be a positive weak solution of (3.1.1) in Q_T. Then for any $y \in \mathbb{R}^N$ the following alternative holds* *(i) either*

$$u(x,t) \geq \lim_{k\to\infty} u_{k\delta_0}(x-y,t) \qquad \forall (x,t) \in Q_T, \tag{3.1.19}$$

(ii) or there exist an open neighborhood U of y and a Radon measure $\mu_U \in \mathfrak{M}_+(U)$ such that

$$\lim_{t\to 0} \int_U u(x,t)\zeta(x)dx = \int_U \zeta d\mu_U \quad \forall \zeta \in C_c(U). \tag{3.1.20}$$

Actually, since (3.1.12) is verified, (3.1.19) is equivalent to the fact that, for any open neighborhood U of y, there holds

$$\limsup_{t\to 0} \int_U u(x,t)dx = \infty. \tag{3.1.21}$$

However, if (3.1.12) is not verified, there only holds (3.1.19) $\Longrightarrow$ (3.1.21).

The set of points y such that (3.1.20) (resp. (3.1.21)) holds is clearly open (resp. closed) and denoted by $\mathcal{R}(u)$ (resp ($\mathcal{S}(u)$). Using a partition of unity, there exists a unique Radon measure $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ such that

$$\lim_{t\to 0} \int_{\mathcal{R}(u)} u(x,t)\zeta(x)dx = \int_{\mathcal{R}(u)} \zeta d\mu \quad \forall \zeta \in C_c(\mathcal{R}(u)). \tag{3.1.22}$$

Owing to the above result we define the *initial trace* of a positive solution u (3.1.1) in Q_T as the couple $(\mathcal{S}(u), \mu)$ for which (3.1.20) and (3.1.21) hold and we denote it by $tr_{_{\mathbb{R}^N}}(u)$. The set $\mathcal{S}(u)$ is the *set of singular points* of $tr_{_{\mathbb{R}^N}}(u)$, while μ is the *regular part* of $tr_{_{\mathbb{R}^N}}(u)$. It is classical that any $\nu \in \mathcal{B}^{reg}(\mathbb{R}^N)$, the set of positive outer regular Borel measures in $\mathbb{R}^N$, can be represented by a couple $(\mathcal{S}, \mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$ and $\mu \in \mathfrak{M}_+(\mathcal{R})$, where $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$, in the following way

$$\nu(A) = \begin{cases} \infty & \text{if } A \cap \mathcal{S} \neq \emptyset, \\ \mu(A) & \text{if } A \subset \mathcal{R}, \end{cases} \qquad \forall A \text{ Borel}.$$

Therefore Theorem 3.1.8 means that $tr_{_{\mathbb{R}^N}}(u) \in \mathcal{B}^{reg}(\mathbb{R}^N)$.

The initial trace can be made more precise when we know whether $\lim_{k\to\infty} u_{k\delta_0}$ is equal to ϕ_∞ or is infinite.

Theorem 3.1.9 *Assume $p \geq 2$ and* (3.1.12) *holds. Let u be a positive solution of* (3.1.1) *in Q_∞.*

I- If $J < \infty$ is verified and $\lim_{k\to\infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ for every $(x,t) \in Q_\infty$. Then either $tr_{_{\mathbb{R}^N}}(u)$ is the Borel measure infinity ν_∞ which satisfies $\nu_\infty(\mathcal{O}) = \infty$ for any non-empty open subset $\mathcal{O} \subset \mathbb{R}^N$, or is a positive Radon measure μ on $\mathbb{R}^N$.

II- If $J = \infty$ is verified and $\lim_{k\to\infty} u_{k\delta_0} = \infty$ in Q_∞. Then $tr_{_{\mathbb{R}^N}}(u)$ is a positive Radon measure μ on $\mathbb{R}^N$

As a consequence of I, there exist infinitely many positive solutions u of (3.1.1) in Q_∞ such that $tr_{_{\mathbb{R}^N}}(u) = \nu_\infty$. By Theorem 3.1.3, Theorem 3.1.4, the previous results apply in particular if $f(s) = s^\alpha \ln^\beta(s+1)$.

3.2 Isolated singularities

Throughout the article c_i denote positive constants depending on N, p, f and sometimes other quantities such as test functions or particular exponents, the value of which may change from one occurrence to another.

3.2.1 The semigroup approach

We refer to [9, p 117] for the detail of the Banach space framework for the construction of solutions of (3.1.1) in Q_∞ with initial data in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We set

$$J(u) = \int_{\mathbb{R}^N} \left(\frac{1}{p} |\nabla u|^p + F(u) \right) dx \tag{3.2.1}$$

when u belongs to the domain $D(J)$ of J which is the set of $u \in L^2(\mathbb{R}^N)$ such that $|\nabla u| \in L^p(\mathbb{R}^N)$ and $F(u) \in L^1(\mathbb{R}^N)$, and $J(u) = \infty$ if $u \notin D(J)$. Then J is a proper convex lower semicontinuous function in $L^2(\mathbb{R}^N)$. Its sub-differential A is defined by its domain $D(A)$ which is the set of $u \in L^2(\mathbb{R}^N)$ such that $|\nabla u| \in L^p(\mathbb{R}^N)$ and $F(u) \in L^1(\mathbb{R}^N)$ with the property that $-\Delta_p u + f(u) \in L^2(\mathbb{R}^N)$ and

$$-\int_{\mathbb{R}^N} v\Delta_p u dx = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u.\nabla v dx \qquad \forall v \in D(J), \tag{3.2.2}$$

and by its expression

$$Au = -\Delta_p u + f(u) \qquad \forall u \in D(A). \tag{3.2.3}$$

Notice that (3.2.2) implies that $vf(u) \in L^1(\mathbb{R}^N)$ for all $v \in D(J)$. The restriction of the operator A is accretive in $L^1(\mathbb{R}^N)$ and in $L^\infty(\mathbb{R}^N)$, hence in $L^q(\mathbb{R}^N)$ for every $q \in [1, \infty]$. The operator A_q defined in $L^q(\mathbb{R}^N)$ is the closure in $L^q(\mathbb{R}^N)$ of the restriction of A to $L^q(\mathbb{R}^N)$. It is a m-accretive operator, with domain $D(A_q)$. Since $C_0^\infty(\mathbb{R}^N) \subset D(A_q)$, $D(A_q)$ is dense in $L^q(\mathbb{R}^N)$. If $u_0 \in L^q$ the generalized solution u to

$$\begin{cases} \dfrac{du}{dt} + A_q u = 0 & \text{in } (0, \infty) \\ \quad u(0) = u_0 & \end{cases} \tag{3.2.4}$$

is obtained by the Crandall-Liggett scheme

$$\frac{u_i - u_{i-1}}{h} + A_q u_i = 0 \qquad i = 0, 1, ... \tag{3.2.5}$$

when we let $h \to 0$, in the sense that the continuous piecewise linear function U_h defined by $U_h(ih) = u_i$ converges to u in the $C([0,T], L^q(\mathbb{R}^N))$-topology, for every $T > 0$. Furthermore, if $q = 2$ and $u_0 \in D(A_2)$ (resp. $u_0 \in L^2(\mathbb{R}^N)$), then $\frac{dU_h}{dt}$ converges to $\frac{du}{dt}$ in $L^2([0,T], L^2(\mathbb{R}^N))$ (resp. $L^2([0,T], L^2(\mathbb{R}^N); tdt)$) (see [20]). We shall denote by $\{S^{A_q}(t)\}_{t>0}$ the semigroup of contractions of $L^q(\mathbb{R}^N)$ generated by $-A_q$ thru the Crandall-Liggett Theorem [4].

An important property [9, Lemma 2] is that if $w \in L^1(\mathbb{R}^N)$ satisfies

$$A_1 w + \sigma w = h \tag{3.2.6}$$

where $\sigma > 0$ and $h \in L^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} A_1 w dx = 0. \tag{3.2.7}$$

Definition 3.2.1 *(i) A function $u \in C([\delta, \infty); L^1(\mathbb{R}^N))$ where $\delta \geq 0$ is a semigroup solution* (3.1.1) *on (δ, ∞) if for any $t \geq \delta$ there holds $u(., t) = S^{A_1}(t - \delta)[u(., \delta)]$.*

(ii) A function $u \in C((\delta, \infty); L^1(\mathbb{R}^N))$ is an extended semigroup solution of (3.1.1) *on (δ, ∞) if for any $t \geq \tau > \delta$, there holds $u(., t) = S^{A_1}(t - \tau)[u(., \tau)]$.*

3.2.2 The Barenblatt-Prattle solutions

We recall the explicit expression, due to Barenblatt and Prattle, of the solution $v = v_{k\delta_0}$ of problem (3.1.14). If $p = 2$

$$v_{k\delta_0}(x, t) = k(4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}, \tag{3.2.8}$$

and if $\frac{2N}{N+1} < p \neq 2$,

$$v_{k\delta_0}(x, t) = t^{-\lambda} V\left(\frac{x}{t^{\frac{\lambda}{N}}}\right), \text{ where } V(\xi) = \left(C_k - d|\xi|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}} \tag{3.2.9}$$

with

$$\lambda = \frac{N}{N(p-2) + p} \qquad \text{and} \qquad d = \frac{p-2}{p}\left(\frac{\lambda}{N}\right)^{\frac{1}{p-1}}, \tag{3.2.10}$$

and where C_k is connected to the mass k by

$$C_k = c(N, p)k^{\ell} \qquad \text{with} \qquad \ell = \frac{p(p-2)\lambda}{(p-1)N}. \tag{3.2.11}$$

The condition $p > \frac{2N}{N+1}$ appears in order that λ be positive. Notice that, if $p > 2$ then $d > 0$, therefore the support of $v_{k\delta_0}(., t)$ is the ball $B_{\delta_k(t)}$ where $\delta_k(t) = \left(\frac{C_k}{d}\right)^{\frac{p-1}{p}} t^{\frac{\lambda}{N}}$, while $v_{k\delta_0}(x, t) > 0$ for all $(x, t) \in Q_\infty$ if $\frac{2N}{N+1} < p < 2$ (and also $p = 2$ although the expression of $v_{k\delta_0}$ is different). Furthermore, if $\frac{2N}{N+1} < p < 2$, the limit of $v_{k\delta_0}$ when $k \to \infty$ is explicit

$$v_\infty(x, t) = \Lambda_N \left(\frac{t}{|x|^p}\right)^{\frac{1}{2-p}}, \tag{3.2.12}$$

where $\Lambda_N = (-d)^{\frac{p-1}{p-2}}$. This type of singular solution which is singular on the whole axis $\{0\} \times (0, \infty) \subset Q_\infty$, is called a *razor blade* (see [19] for some examples). To this solution corresponds a universal estimate.

Lemma 3.2.2 *Assume $1 < p < 2$ and let $v \in C(\overline{Q_\infty} \setminus B_{R_0} \times \{0\})$ be a semigroup solution positive of* (3.1.1)

$$\partial_t v - \Delta_p v = 0 \qquad \text{in } Q_\infty \tag{3.2.13}$$

which satisfies

$$\lim_{t\to 0}\int_K v(x,t)dx = 0, \tag{3.2.14}$$

for any compact set $K \subset \mathbb{R}^N \setminus B_{R_0}$. *Then there exists* $c_1 = c_1(N,p) > 0$ *such that*

$$\sup_{0\le\tau\le t}\int_{\{x:|x|>R\}} v(x,\tau)dx \le c_1\left(\frac{t}{(R-R_0)^{\frac{N}{\lambda}}}\right)^{\frac{1}{2-p}} \qquad \forall R > R_0, t > 0. \tag{3.2.15}$$

If we assume moreover that $\lim_{|x|\to\infty} v(x,t) = 0$ *locally uniformly with respect to* $t \ge 0$, *then*

$$v(x,t) \le \Lambda_1\left(\frac{t}{(|x|-R_0)^p}\right)^{\frac{1}{2-p}} \qquad \forall (x,t) \in Q_\infty, |x| > R_0, \tag{3.2.16}$$

where Λ_1 *is the value of the constant in* (3.2.12) *when* $N = 1$.

Proof. The first estimate is a consequence of

$$\sup_{0\le\tau\le t}\int_{B_\rho(a)} v(x,t)dx \le c_2\left(\int_{B_{2\rho}(a)} v(x,0)dx + \left(\frac{t}{\rho^{\frac{N}{\lambda}}}\right)^{\frac{1}{2-p}}\right) \tag{3.2.17}$$

in [6, Lemma III.3.1] under the assumption that $v(.,0)$ is continuous with compact support. Actually this assumption is not used. In this proof the first step is the following estimate obtained by a suitable choice of test function :

$$\sup_{0\le\tau\le t}\int_{B_R(a)} v(x,t)dx \le \int_{B_{2R}(a)} v(x,0)dx + \frac{c_3}{R}\int_0^t\int_{B_R(a)} |\nabla v|^{p-1}dx\,d\tau \tag{3.2.18}$$

valid for any $a \in \mathbb{R}^N\setminus\{(0,0)\}$ and $R \le |a|/2$. The second step to get (3.2.17) is to estimate the integral on the right-hand side by relation (I.4.2) in [6, Lemma I.4.1] with the same choice of ϵ. We apply estimate (3.2.17) with a sequence of points in a fixed direction $\mathbf{e}$ (with $|\mathbf{e}| = 1$) $a = a_k = \left(2^k(R-R_0)+R_0\right)\mathbf{e}$ and $\rho = \rho_k = 2^{k-1}(R-R_0)$ (actually we start with $\rho < \rho_k$ and let it grow up to ρ_k). Then we get

$$\sup_{0\le\tau\le t}\int_{B_{\rho_k}(a_k)} v(x,t)dx \le c_4 2^{-\frac{N(k-1)}{\lambda(2-p)}}\left(\frac{t}{(R-R_0)^{\frac{N}{\lambda}}}\right)^{\frac{1}{2-p}}. \tag{3.2.19}$$

Since the ball $B_{\rho_k}(a_k)$ and $B_{\rho_{k+1}}(a_{k+1})$ are overlapping there exist a finite number of points $\{\mathbf{e}_j\}_{j=1}^{d_1}$ and $\{\mathbf{e}'_j\}_{j=1}^{d_2}$ (d_1 and d_2 depend only on N) on the unit sphere such that

$$\left\{x\in\mathbb{R}^N : |x| \ge R\right\} \subset \left(\bigcup_{j=1}^{d_1}\bigcup_{k=1}^{\infty} B_{\rho_k}(2\rho_k\mathbf{e}_j)\right)\bigcup\left(\bigcup_{j=1}^{d_2} B_{\frac{R-R_0}{2}}(R\mathbf{e}'_j)\right).$$

Therefore

$$\sup_{0\le\tau\le t}\int_{\{x:|x|>R\}} v(x,\tau)dx \le c_4\left[d_1\sum_{k=0}^{\infty}2^{-\frac{Nk}{\lambda(2-p)}} + d_2 2^{\frac{N}{\lambda(2-p)}}\right]\left(\frac{t}{(R-R_0)^{\frac{N}{\lambda}}}\right)^{\frac{1}{2-p}}$$

which is (3.2.15).

Estimate (3.2.16) follows from comparison with the 1-dim form of v_∞

$$v_\infty(s,t) = \Lambda_1 \left(\frac{t}{s^p}\right)^{\frac{1}{2-p}} \qquad \forall s,t>0. \tag{3.2.20}$$

For $\epsilon > 0$, the function

$$(x,t) \mapsto v_\infty(x_1 - R_0 - \epsilon, t) + \epsilon$$

where $x = (x_1, ..., x_N) = (x_1, x')$, is a solution of (3.2.13) in $H_{1,R_0+\epsilon} \times (0,\infty)$ where $H_{1,m} = \{x \in \mathbb{R}^N : x_1 > m\}$. For R large enough $v(x,t) \le v_\infty(x_1 - R_0 - \epsilon, t) + \epsilon$ on the set $((H_{1,R_0+\epsilon} \cap \partial B_R) \cup (\partial H_{1,R_0+\epsilon} \cap B_R)) \times [0,T]$ for any $T > 0$, and for $t = 0$. By the maximum principle $v(x,t) \le v_\infty(x_1 - R_0 - \epsilon, t) + \epsilon$ in $(H_{1,R_0+\epsilon} \cap B_R) \times (0,T]$. Letting successively $R \to \infty$, $T \to \infty$ and $\epsilon \to 0$ yields

$$v(x,t) \le v_\infty(x_1 - R_0, t) \quad \forall (x,t) \in H_{1,R_0} \times (0,\infty).$$

Next we take $(x,t) \in Q_\infty$ such that $|x| > R_0$ and consider a rotation $\mathcal{R}_x$ such that $\mathcal{R}_x(x) = X := (|x|, 0, ..., 0)$. Set $v_{\mathcal{R}_x}(y,t) = v(\mathcal{R}_x^{-1}(y), t)$ for every $(y,t) \in Q_\infty$, $|y| > R_0$. Since the equation (3.2.13) is invariant under the rotation $\mathcal{R}_x$, $v_{\mathcal{R}_x}$ is also a solution of (3.2.13) and satisfies (3.2.14) and $\lim_{|x|\to\infty} v_{\mathcal{R}_x}(x,t) \to 0$ locally uniformly with respect to $t \ge 0$. Hence

$$v_{\mathcal{R}_x}(y,t) \le v_\infty(y_1 - R_0, t) \quad \forall (y,t) \in H_{1,R_0} \times (0,\infty).$$

On one hand, by taking $y = X$, we get

$$v_{\mathcal{R}_x}(X,t) \le v_\infty(|x| - R_0, t). \tag{3.2.21}$$

On the other hand,

$$v_{\mathcal{R}_x}(X,t) = v(\mathcal{R}_x^{-1}(\mathcal{R}_x(x)), t) = v(x,t). \tag{3.2.22}$$

Thus (3.2.16) follows from (3.2.21) and (3.2.22). □

Proposition 3.2.3 *Assume $p > \frac{2N}{N+1}$ and let $\{v^n\} \subset C([0,\infty); L^1(\mathbb{R}^N))$ be a sequence of positive semigroup solutions of (3.2.13) on $(0,\infty)$ such that $v^n(.,0)$ has support in B_{ϵ_n} where $\epsilon_n \to 0$. If*

$$\int_{\mathbb{R}^N} v^n(x,0)dx = k_n \to k \quad \text{as } n \to \infty$$

then $v^n \to v_{k\delta_0}$ locally uniformly in Q_∞.

Proof. We first give the proof in the case $\frac{2N}{N+1} < p < 2$. By *a priori* estimates, up to a subsequence, $\{v^n\}$ converges locally uniformly in Q_∞ to a solution v of (3.2.13) in Q_∞. By Herrero-Vázquez mass conservation property [9, Theorem 2] (valid if $p > \frac{2N}{N+1}$)

$$\int_{\mathbb{R}^N} v^n(x,t)dx = \int_{\mathbb{R}^N} v^n(x,0)dx = k_n.$$

By (3.2.16)

$$v^n(x,t) \le \Lambda_1 \left(\frac{t}{(|x| - \epsilon_n)^p} \right)^{\frac{1}{2-p}} \qquad \forall t > 0,\ \forall |x| > \epsilon_n.$$

Since $\frac{p}{2-p} > N$, the function

$$x \mapsto \left(\frac{t}{(|x| - \epsilon_n)^p} \right)^{\frac{1}{2-p}}$$

belongs to $L^1(\mathbb{R}^N \setminus B_\delta)$, for any $\delta > \epsilon_n$. Since $v^n(x,t) \to v(x,t)$ uniformly in B_δ, it follows by the dominated convergence theorem

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} v^n(x,t)dx = \int_{\mathbb{R}^N} v(x,t)dx = k. \tag{3.2.23}$$

Because v is a positive solution with isolated singularity at $(0,0)$, it follows from [3] that $v = v_{k\delta_0}$, solution of (3.1.14).

When $p \ge 2$, the function $v_{k\delta_0}(.,t)$ has a compact support $D_{k_n}(t)$ for any $t > 0$ and $D_{k_n}(t) \subset B_{R_n(t)}$ where

$$R_n(t) = \epsilon_n + c_5 k_n^{\frac{p-2}{p}}\, t^{\frac{1}{N(p-2)+p}} \le \epsilon^* + c_5 k_*^{\frac{p-2}{p}}\, t^{\frac{1}{N(p-2)+p}} \tag{3.2.24}$$

where $c_5 = c_5(N,p) > 0$, $\epsilon^* = \sup\{\epsilon_n; n \in \mathbb{N}\}$ and $k_* = \sup\{k_n; n \in \mathbb{N}\}$. Using Lebesgue dominating theorem we obtain again (3.2.23). □

3.2.3 Fundamental solutions

The following lemma is fundamental.

Lemma 3.2.4 *Assume $p > \frac{2N}{N+1}$. Then, for any $k, R, T > 0$,*

$$\int_1^\infty f(s)s^{-\frac{p(N+1)}{N}} ds < \infty \Longrightarrow f(v_{k\delta_0}) \in L^1(B_R \times (0,T)). \tag{3.2.25}$$

Proof. The result is already proved in the chapitre 2 for the case $p = 2$ and in [10] for the case $p > 2$. It appears to be new in the case $\frac{2N}{N+1} < p < 2$. Without any loss of generality we can assume $R = T = 1$ and we consider $\frac{2N}{N+1} < p < 2$. We set $d^* = -d$. By rescaling we can assume that $C_k = d^* = 1$. Therefore

$$I = \int\int_{B_1\times(0,1)} f(v_{k\delta_0})dxdt = \omega_N \int_0^1 \int_0^1 f\left(t^{-\lambda} \left[1 + \left(\frac{r}{t^{\frac{\lambda}{N}}} \right)^{\frac{p}{p-1}} \right]^{\frac{p-1}{p-2}} \right) r^{N-1} dr\, dt.$$

Set $s = t^{-\lambda} \left[1 + \left(\frac{r}{t^{\frac{\lambda}{N}}} \right)^{\frac{p}{p-1}} \right]^{\frac{p-1}{p-2}}$, then $r = t^{\frac{\lambda}{N}} \left[(t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right]^{\frac{p-1}{p}}$ and

$$\begin{aligned} I &= \tfrac{2-p}{p}\omega_N \int_0^1 \int_{t^{-\lambda}(1+t^{-\frac{\lambda p}{p-1}})^{\frac{p-1}{p-2}}}^{t^{-\lambda}} (t^\lambda s)^{-\frac{1}{p-1}} \left((t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right)^{\frac{N(p-1)}{p}-1} f(s)ds\, t^{2\lambda}dt \\ &= \tfrac{2-p}{p}\omega_N (I_1 + I_2) \end{aligned}$$

where

$$I_1 = \int_{2^{\frac{p-1}{p-2}}}^{1} \int_{a(s)}^{1} (t^\lambda s)^{-\frac{1}{p-1}} \left((t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right)^{\frac{N(p-1)}{p}-1} t^{2\lambda} dt\, f(s) ds$$

$$I_2 = \int_{1}^{\infty} \int_{a(s)}^{s^{-\frac{1}{\lambda}}} (t^\lambda s)^{-\frac{1}{p-1}} \left((t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right)^{\frac{N(p-1)}{p}-1} t^{2\lambda} dt\, f(s)$$

and $a(s)$ is the inverse function of $t \mapsto t^{-\lambda}(1+t^{-\frac{\lambda p}{p-1}})^{\frac{p-1}{p-2}}$. Clearly

$$t^{-\lambda}(1+t^{-\frac{\lambda p}{p-1}})^{\frac{p-1}{p-2}} \le t^{\frac{2\lambda(p-1)}{2-p}} \Longrightarrow a(s) \ge s^{\frac{2-p}{2\lambda(p-1)}}.$$

Therefore

$$\begin{aligned} I_1 &\le \int_{2^{\frac{p-1}{p-2}}}^{1} \int_{s^{\frac{2-p}{2\lambda(p-1)}}}^{1} (t^\lambda s)^{-\frac{1}{p-1}} \left((t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right)^{\frac{N(p-1)}{p}-1} t^{2\lambda} dt\, f(s) ds \\ &\le \frac{1}{\lambda} \int_{2^{\frac{p-1}{p-2}}}^{1} \int_{s^{\frac{p}{2(p-1)}}}^{s} (1-\tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{\frac{1}{\lambda}+\frac{N(p-2)}{p}} d\tau\, s^{-2-\frac{1}{\lambda}} f(s) ds. \end{aligned}$$

Since $\frac{1}{\lambda} + \frac{N(p-2)}{p} > -1$ and $\frac{N(p-1)}{p} - 1 > -1$,

$$\int_{s^{\frac{p}{2(p-1)}}}^{s} (1-\tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{\frac{1}{\lambda}+\frac{N(p-2)}{p}} d\tau < \int_0^1 (1-\tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{\frac{1}{\lambda}+\frac{N(p-2)}{p}} d\tau < \infty.$$

Furthermore $-2 - \frac{1}{\lambda} = -p - \frac{p}{N}$ thus

$$I_1 \le c_6 \int_{2^{\frac{p-1}{p-2}}}^{1} f(s) s^{-\frac{p(N+1)}{N}} ds.$$

We perform the same change of variable with I_2

$$\begin{aligned} I_2 &\le \int_{1}^{\infty} \int_{s^{\frac{2-p}{2\lambda(p-1)}}}^{s^{-\frac{1}{\lambda}}} (t^\lambda s)^{-\frac{1}{p-1}} \left((t^\lambda s)^{\frac{p-2}{p-1}} - 1 \right)^{\frac{N(p-1)}{p}-1} t^{2\lambda} dt\, f(s) ds \\ &\le \frac{1}{\lambda} \int_{1}^{\infty} \int_{s^{\frac{p}{2(p-1)}}}^{1} (1-\tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{\frac{1}{\lambda}+\frac{N(p-2)}{p}} d\tau\, s^{-2-\frac{1}{\lambda}} f(s) ds. \end{aligned}$$

Again

$$\int_{s^{\frac{p}{2(p-1)}}}^{1} (1-\tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{1+\frac{N(p-2)}{p}} d\tau < \int_0^1 (1-\tau^{\frac{2-p}{p-1}})^{\frac{N(p-1)}{p}-1} \tau^{\frac{1}{\lambda}+\frac{N(p-2)}{p}} d\tau < \infty,$$

then

$$I_2 \le c_7 \int_1^\infty f(s) s^{-\frac{p(N+1)}{N}} ds.$$

Therefore (3.2.25) holds. □

Notice that the assumption implies that $v_{k\delta_0} \in C(Q_\infty) \cap L^\infty(\delta, \infty; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ for every $\delta > 0$.

Proof of Theorem 3.1.1. *Existence.* Let $\epsilon > 0$, $Q_{\epsilon,\infty} = \mathbb{R}^N \times (\epsilon, \infty)$ and denote by u_ϵ the solution of

$$\begin{cases} \partial_t u - \Delta_p u + f(u) = 0 & \text{in } Q_{\epsilon,\infty} \\ u(.,\epsilon) = v_{k\delta_0}(.,\epsilon) & \text{in } \mathbb{R}^N. \end{cases} \tag{3.2.26}$$

Since $v_{k\delta_0}(.,\epsilon)$ is a smooth positive function belonging to $L^1(\mathbb{R}^N)$ the function u_ϵ is constructed by truncation. By the maximum principle

$$u_\epsilon(x, t+\epsilon) \leq v_{k\delta_0}(x, t+\epsilon) \qquad \forall (x,t) \in Q_{\epsilon,\infty}. \tag{3.2.27}$$

For $0 < \epsilon' < \epsilon$, $u_{\epsilon'}(x,\epsilon) \leq v_{k\delta_0}(x,\epsilon) = u_\epsilon(x,\epsilon)$, thus $u_{\epsilon'}(x, t+\epsilon) \leq u_\epsilon(x, t+\epsilon)$ in $Q_{\epsilon,\infty}$. Set $\tilde{u} = \lim_{\epsilon \to 0} u_\epsilon$, then $\tilde{u} \leq v_{k\delta_0}$ in Q_∞. By the standard local regularity theory for degenerate equations, ∇u_ϵ remains locally compact in $(C^1_{loc}(Q_\infty))^N$, thus $\tilde{u}$ satisfies (3.1.1) in Q_∞.

In order to prove that

$$\frac{d}{dt}\int_{\mathbb{R}^N} u_\epsilon(x,s)dx + \int_{\mathbb{R}^N} f(u_\epsilon(x,s))dx = 0$$

we recall that u_ϵ can be obtained as the limit of thru the iterative implicit scheme (3.2.4) with $q \in [1,\infty]$ is arbitrary since $u_{\epsilon,0} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For $h > 0$ we can write it under the form

$$u_{\epsilon,i} - h\Delta_p u_{\epsilon,i} = -hf(u_{\epsilon,i}) + u_{\epsilon,i-1}.$$

By (3.2.7), and denoting by $\tilde{U}_{\epsilon,h}$ the piecewise constant function such that $\tilde{U}_{\epsilon,h}(jh) = u_{\epsilon,j}$, we obtain since $u_{\epsilon,0} = v_{k\delta_0}(\epsilon)$

$$\int_{\mathbb{R}^N} (u_{\epsilon,i} - v_{k\delta_0}(\epsilon))(x)dx = -\int_\epsilon^{ih}\int_{\mathbb{R}^N} f(\tilde{U}_{\epsilon,h}(x))dxdt. \tag{3.2.28}$$

Letting $h \to 0$ and $i \to \infty$ such that $ih = t > \epsilon$ and using the uniform convergence, we obtain

$$\int_{\mathbb{R}^N} u_\epsilon(x,t)dx - \int_{\mathbb{R}^N} v_{k\delta_0}(x,\epsilon)dx = -\int_\epsilon^t\int_{\mathbb{R}^N} f(u_\epsilon(x,s))dxdt. \tag{3.2.29}$$

Since $0 \leq u_\epsilon \leq v_{k\delta_0}$ and $v_{k\delta_0}(.,t)$ has constant mass equal to k, we derive

$$\left|\int_{\mathbb{R}^N} u_\epsilon(x,t)dx - k\right| \leq \int_\epsilon^t\int_{\mathbb{R}^N} f(v_{k\delta_0}(x,s))dxdt. \tag{3.2.30}$$

Because $f(v_{k\delta_0}) \in L^1(\mathbb{R}^N \times (0,T))$, we can let $\epsilon \to 0$, using the monotone convergence theorem, in order to get

$$\left|\int_{\mathbb{R}^N} u(x,t)dx - k\right| \leq \int_0^t\int_{\mathbb{R}^N} f(v_{k\delta_0}(x,s))dxdt. \tag{3.2.31}$$

This implies that

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(x,t)dx = k. \tag{3.2.32}$$

If $\phi \in C_c(\mathbb{R}^N)$, let $\zeta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \zeta \leq 1$, $\zeta = 1$ on the support of ϕ and $\zeta(0) = 1$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} u(x,t)\phi(x)dx &= \int_{\mathbb{R}^N} u(x,t)\phi(x)\zeta(x)dx \\ &= \phi(0)\int_{\mathbb{R}^N} u(x,t)dx + \int_{\mathbb{R}^N} u(x,t)(\phi(x)\zeta(x) - \phi(0))dx.\end{aligned}$$

Thus

$$\left|\int_{\mathbb{R}^N} u(x,t)\phi(x)dx - \phi(0)\int_{\mathbb{R}^N} u(x,t)dx\right| \leq \int_{\mathbb{R}^N} v_{k\delta_0}(x,t)\left|\phi(x)\zeta(x) - \phi(0)\right| dx.$$

Because $|\phi(x)\zeta(x) - \phi(0)|$ is continuous and vanishes at zero and $v_{k\delta_0}(.,0) = k\delta_0$, it follows from (3.2.32)

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(x,t)\phi(x)dx = k\phi(0). \tag{3.2.33}$$

Uniqueness. The proof uses some ideas from [10, Th 2.4]. Assume $\tilde{u}$ is any nonnegative solution of problem (3.1.13), then, for any $\epsilon > 0$ we denote by $\tilde{v}_\epsilon$ the solution of

$$\left\{\begin{array}{ll} \partial_t v - \Delta_p v = 0 & \text{in } Q_{\epsilon,\infty} \\ v(.,\epsilon) = \tilde{u}(.,\epsilon) & \text{in } \mathbb{R}^N. \end{array}\right. \tag{3.2.34}$$

By the maximum principle $\tilde{v}_\epsilon \geq \tilde{u}$ in $Q_{\epsilon,\infty}$. When $\epsilon \to 0$, $\tilde{v}_\epsilon$ converges locally uniformly to a solution $\tilde{v}$ of the same equation in Q_∞. Furthermore, using again [9, Lemma 2],

$$\int_{\mathbb{R}^N} \tilde{v}_\epsilon(x,t+\epsilon)dx = \int_{\mathbb{R}^N} \tilde{u}(x,\epsilon)dx.$$

By Fatou's Lemma and using the fact that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^N} \tilde{u}(x,\epsilon)dx = k,$$

we derive

$$\int_{\mathbb{R}^N} \tilde{v}(x,t)dx \leq k. \tag{3.2.35}$$

Since $\tilde{v} \geq \tilde{u}$, equality holds in (3.2.35). Since the fundamental solution is unique [3, Th 4.1], it implies $\tilde{v} = v_{k\delta_0}$ and $\tilde{u} \leq v_{k\delta_0}$. We end the proof as in [3, Th 4.1], using the L^1-contraction mapping principle and the fact that any solution of (3.1.13) is smaller than $v_{k\delta_0}$: for $t > s > 0$, there holds

$$\begin{aligned}\int_{\mathbb{R}^N} |u(x,t) - \tilde{u}(x,t)|\, dx &\leq \int_{\mathbb{R}^N} |u(x,s) - \tilde{u}(x,s)|\, dx \\ &\leq \int_{\mathbb{R}^N} |u(x,s) - v_{k\delta_0}(x,s)|\, dx + \int_{\mathbb{R}^N} |v_{k\delta_0}(x,s) - \tilde{u}(x,s)|\, dx \\ &\leq \int_{\mathbb{R}^N} (v_{k\delta_0}(x,s) - u(x,s))dx + \int_{\mathbb{R}^N} (v_{k\delta_0}(x,s) - \tilde{u}(x,s))dx.\end{aligned} \tag{3.2.36}$$

When $s \to 0$ the right-hand side of the last line goes to 0. This implies the claim. □

The next result shows some geometric properties of the $u_{k\delta_0}$.

Proposition 3.2.5 *The solution $u = u_{k\delta_0}$ of problem* (3.1.15) *is radial and nonincreasing with respect to $|x|$.*

Proof. It is sufficient to prove the result with the approximation $u_\epsilon(.,t)$. By (3.2.9), $v_{k\delta_0}(.,t)$ is radial and decreasing, therefore $u_\epsilon(.,t)$ is radial too by uniqueness. We notice that u_ϵ is the increasing limit, when $R \to \infty$, of the solution $u_{\epsilon,R}$ of

$$\begin{cases} \partial_t u - \Delta_p u + f(u) = 0 & \text{in } Q^{B_R}_{\epsilon,\infty} \\ u = 0 & \text{on } \partial B_R \times (\epsilon, \infty) \\ u(.,\epsilon) = v_{k\delta_0}(.,\epsilon) & \text{in } B_R. \end{cases} \tag{3.2.37}$$

For $\lambda \in (0, R)$, we set $\Sigma_\lambda = B_R \cap \{x = (2\lambda - x_1, x') : x_1 > \lambda\} \cap B_R$ and define w_λ by

$$w_\lambda(x,t) = w_\lambda(x_1, x', t) := u_{\lambda,\epsilon,R}(x) - u_{\epsilon,R}(x) = u_{\epsilon,R}(2\lambda - x_1, x', t) - u_{\epsilon,R}(x_1, x', t).$$

If $Q^{\Sigma_\lambda}_{\epsilon,\infty} = \Sigma_\lambda \times (\epsilon, \infty)$, there holds

$$\begin{cases} \partial_t w_\lambda + \mathcal{A} w_\lambda + d(x) w_\lambda = 0 & \text{in } Q^{\Sigma_\lambda}_{\epsilon,\infty} \\ w_\lambda \geq 0 & \text{in } \partial\Sigma_\lambda \times (\epsilon, \infty) \\ w_\lambda(.,\epsilon) \geq 0 & \text{in } \Sigma_\lambda. \end{cases} \tag{3.2.38}$$

where

$$d(x) = \begin{cases} \frac{f(u_{\lambda,\epsilon,R}) - f(u_{\epsilon,R})}{u_{\lambda,\epsilon,R} - u_{\epsilon,R}} & \text{if } u_{\lambda,\epsilon,R} \neq u_{\epsilon,R} \\ 0 & \text{if } u_{\lambda,\epsilon,R} = u_{\epsilon,R} \end{cases}$$

and

$$\mathcal{A} w_\lambda = -\Delta_p u_{\lambda,\epsilon,R} + \Delta_p u_{\epsilon,R}.$$

Notice that $d \geq 0$ since f is nondecreasing and $\mathcal{A}$ is elliptic [7, Lemma 1.3]. Furthermore the boundary data are continuous, therefore $w_\lambda \geq 0$. Letting $\lambda \to 0$, changing λ by $-\lambda$ and replacing the x_1 direction, by any direction going thru 0, we derive that $u_{\epsilon,R}(.,t)$ is radially decreasing. Letting $R \to \infty$ yields to $u_\epsilon(.,t)$ is radially decreasing too. □

In the next result we characterize positive solutions of (3.1.1) with an isolated singularity at $t = 0$

Proposition 3.2.6 *Assume $p > \frac{2N}{N+1}$ and f satisfies* (3.1.12). *If $u \in C(\overline{Q_\infty} \setminus \{(0,0)\})$ is a positive semigroup solution of* (3.1.1) *in Q_∞ such that $u(x,0) = 0$, for all $x \neq 0$ and*

$$\lim_{t \to 0} \int_{\mathbb{R}^N} u(x,t)dx < \infty,$$

then there exists $k \geq 0$ such that $u = u_{k\delta_0}$.

Proof. Using [11, Lemma 2.2] when $p \geq 2$, or the proof of Theorem 3.1.1 when $\frac{2N}{N+1} < p < 2$ jointly with the fact that

$$t \mapsto \int_{\mathbb{R}^N} u(x,t)dx$$

is decreasing, we derive that $u \leq v_{m\delta_0}$ for some $m \geq 0$ and there exists $k \geq 0$ such that

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(x,t)dx = k.$$

Since $u(.,0)$ vanishes if $x \neq 0$, it implies

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(x,t)\phi(x)dx = k\phi(0) \qquad \forall \phi \in C_c(\mathbb{R}^N).$$

Therefore u satisfies (3.1.13). By uniqueness, $u = u_{k\delta_0}$. □

3.2.4 Strong singularities

This section is devoted to study the limit of the sequence of the solutions $u_{k\delta_0}$ to (3.1.13) as $k \to \infty$ with $f(s) = s^\alpha \ln^\beta(s+1)$ where $p > 2$, $\alpha \in [1, p-1)$ and $\beta > 0$.

Proof of Theorem 3.1.3. By the comparison principle,

$$u_{k\delta_0}(x,t) \leq v_{k\delta_0}(x,t) \leq c_8 k^{\frac{(p-1)\ell}{p-2}} t^{-\lambda},$$

where $v_{k\delta_0}$ is the solution of (3.1.14) in Q_∞ and $c_8 = c_8(N,p) > 0$ in (3.2.11). We set

$$\theta_k(t) = c_8^{\alpha-1} k^{\frac{\ell(\alpha-1)(p-1)}{p-2}} t^{-\lambda(\alpha-1)} \ln^\beta(c_8 k^{\frac{(p-1)\ell}{p-2}} t^{-\lambda} + 1) \tag{3.2.39}$$

then

$$\partial_t u_{k\delta_0} - \Delta_p u_{k\delta_0} + u_{k\delta_0}\theta_k(t) \geq 0. \tag{3.2.40}$$

Next we write $u_{k\delta_0}(x,t) = b_k(t)w_k(x, s_k(t))$ (the functions b_k and s_k will be defined later). For simplicity, we drop the subscript k in b_k and s_k. Inserting in (3.2.40), we get

$$b^{2-p}(t)s'(t)\partial_s w_k(x,s) - \Delta_p w_k(x,s) + b^{1-p}[b'(t) + b(t)\theta_k(t)]w_k(x,s) \geq 0. \tag{3.2.41}$$

We choose the functions b and s such that

$$b^{2-p}(t)s'(t) = 1 \qquad \text{and} \qquad b'(t) + b(t)\theta_k(t) = 0,$$

which implies

$$b(t) = \exp\big(-\int_0^t \theta_k(\tau)d\tau\big) \quad \text{and} \quad s(t) = \int_0^t \exp\big(-(p-2)\int_0^\tau \theta_k(\sigma)d\sigma\big)d\tau. \tag{3.2.42}$$

Then $\partial_s w_k - \Delta_p w_k \geq 0$ in $\mathbb{R}^N \times (0, s_{k,0})$ with some $s_{k,0} > 0$ and $w_k(.,0) = k\delta_0$. It follows by comparison principle that $w_k \geq v_{k\delta_0}$ in $\mathbb{R}^N \times (0, s_{k,0})$. Hence

$$u_{k\delta_0}(x,t) \geq b(t)v_{k\delta_0}(x,s) = \exp\left(-\int_0^t \theta_k(\tau)d\tau\right) s^{-\lambda}\big(c_9 k^\ell - c_{10} s^{\frac{-p\lambda}{(p-1)N}} |x|^{\frac{p}{p-1}}\big)_+^{\frac{p-1}{p-2}}. \tag{3.2.43}$$

Let $\delta_1 > \frac{\ell(\alpha-1)(p-1)}{p-2}$ and $0<\delta_2<1-\lambda(\alpha-1)$. Using (3.2.39) there exists $t_0>0$ depending on δ_1, δ_2 and k large enough, such that, for any $t\in(0,t_0)$ there holds

$$\int_0^t \theta_k(\tau)d\tau \le c_{11}k^{\delta_1}t^{\delta_2} \tag{3.2.44}$$

with $c_{11}=c_{11}(c_i,\alpha,\beta,p,N)>0$. It follows from (3.2.42) and (3.2.44) that

$$t\exp\left[-(p-2)c_{11}k^{\delta_1}t^{\delta_2}\right]\le s(t)\le t. \tag{3.2.45}$$

Since $J<\infty$ holds, there exists the solution ϕ_∞ of (3.1.2). The sequence $\{u_{k\delta_0}\}$ is increasing and is bounded from above by ϕ_∞, then the function $\underline{U}(x,t):=\lim\limits_{k\to\infty}u_{k\delta_0}(x,t)$ satisfies $\underline{U}(x,t)\le\phi_\infty(t)$ for every $(x,t)\in Q_\infty$. We restrict $x\in B_1$ and we choose t such that

$$c_9k^\ell - c_{10}s(t)^{\frac{-p\lambda}{(p-1)N}} > \frac{1}{2}c_9k^\ell \Longleftrightarrow k>\left(\frac{2c_{10}}{c_9}\right)^{\frac{1}{\ell}}s(t)^{-\frac{1}{p-2}}. \tag{3.2.46}$$

By (3.2.45), we only need to choose t such that

$$k\ge\left(\frac{2c_{10}}{c_9}\right)^{\frac{1}{\ell}}t^{\frac{-1}{p-2}}\exp\left(c_{11}k^{\delta_1}t^{\delta_2}\right). \tag{3.2.47}$$

We choose t under the form

$$t=k^{-\frac{1}{\gamma}} \text{ with } \gamma>0, \tag{3.2.48}$$

then (3.2.47) becomes

$$t^{-\gamma}\ge\left(\frac{2c_{10}}{c_9}\right)^{\frac{1}{\ell}}t^{\frac{-1}{p-2}}\exp\left(c_{11}t^{\delta_2-\delta_1\gamma}\right). \tag{3.2.49}$$

In order to obtain (3.2.49), it is sufficient to choose γ such that

$$\frac{1}{p-2}<\gamma<\frac{\delta_2}{\delta_1}. \tag{3.2.50}$$

Indeed, since $\alpha<p-1$, we may choose δ_1 and δ_2 close enough $\frac{\ell(\alpha-1)(p-1)}{p-2}$ and $1-\lambda(\alpha-1)$ respectively such that (3.2.50) holds true. When t has the form (3.2.48) where γ satisfies (3.2.50), from (3.2.43), (3.2.44)-(3.2.46) and the fact that $\underline{U}\ge u_{k\delta_0}$ in Q_∞, we deduce that

$$\underline{U}(x,t)\ge c_{12}t^{-\lambda}\exp\left[c_{13}\ln(t^{-1})-c_{11}t^{\delta_2-\delta_1\gamma}\right] \tag{3.2.51}$$

for every $(x,t)\in B_1\times(0,t_0)$ with t_0 small enough and $c_{12}=c_{12}(N,p)$, $c_{13}=c_{13}(N,p,\gamma)$. Since γ satisfies (3.2.50),

$$c_{13}\ln(t^{-1})-c_{11}t^{\delta_2-\delta_1\gamma}>0$$

for every $t\in(0,t_0)$. Therefore $\lim_{t\to0}\underline{U}(x,t)=\infty$ uniformly with respect to $x\in B_1$. We next proceed as in [19, Lemma 3.1] to deduce that $\underline{U}(x,t)$ is independent of x and therefore it is a solution of (3.1.2). Since $J<\infty$, $\underline{U}(x,t)=\phi_\infty(t)$ for every $(x,t)\in Q_\infty$. □

Theorem 3.1.4 is proved by the same arguments as Theorem 3.1.3, using the fact that $\underline{U}(x,t)$ is independent of x.

3.3 Non-uniqueness

The next result shows that $K = \infty$ is the necessary and sufficent condition so that a local solution of

$$(r^{N-1}|w'|^{p-2}w')' = r^{N-1}f(w) \tag{3.3.1}$$

can be continued as a global solution. More precisely,

Lemma 3.3.1 *Every positive and increasing solution of* (3.3.1) *defined in an interval* $[a, a^*]$ *to the right of* $a > 0$ *can be continued as a solution of* (3.3.1) *on* $[a, +\infty)$ *if and only if* f *satisfies*

$$\int_\lambda^\infty \frac{ds}{F(s)^{\frac{1}{p}}} = \infty \tag{3.3.2}$$

for any $\lambda > 0$.

Proof. The proof is is an extension to the case $p \neq 2$ of the one of [18, Lemma 2.1] for the case $p = 2$.

Step 1. We first assume that w is defined on a maximal interval $[a, a^*)$ with $a^* < \infty$ and $\lim_{r \to a^*} w(r) = +\infty$. Since w is a nondecreasing function, $w' \geq 0$. And hence we may write (3.3.1) under the following form

$$\frac{N-1}{r}(w')^{p-1} + (p-1)(w')^{p-2}w'' = f(w),$$

which implies that

$$(p-1)(w')^{p-2}w'' \leq f(w) \tag{3.3.3}$$

and hence

$$\frac{p-1}{p}(w'^p)' \leq (F(w))'.$$

Taking the integral over $[a, r]$, we get

$$\frac{p-1}{p}[(w')^p(r) - (w')^p(a)] \leq F(w(r)) - F(w(a)) \leq F(w(r)).$$

Since f is positive on $(0, \infty)$, $F(s) \to \infty$ when $s \to \infty$, thus there exists $\tilde{a} \in (a, a^*)$ such that

$$0 < w'(r)^p \leq \frac{2p}{p-1}F(w(r)) \Longrightarrow \frac{w'(r)}{F(w(r))^{\frac{1}{p}}} \leq \Big(\frac{2p}{p-1}\Big)^{\frac{1}{p}} \qquad \forall r \in [\tilde{a}, a^*).$$

Taking the integral over $[\tilde{a}, r]$, we obtain

$$\int_{w(\tilde{a})}^{w(r)} \frac{ds}{F(s)^{\frac{1}{p}}} \leq \Big(\frac{2p}{p-1}\Big)^{\frac{1}{p}}(r - \tilde{a}).$$

Letting $r \to a^*$ yields to

$$\int_{w(\tilde{a})}^{\infty} \frac{ds}{F(s)^{\frac{1}{p}}} \leq \Big(\frac{2p}{p-1}\Big)^{\frac{1}{p}}(a^* - \tilde{a}) < \infty$$

and (3.3.2) is not satisfied.

Step 2. We assume that

$$\int_\lambda^\infty \frac{ds}{F(s)^{\frac{1}{p}}} < \infty$$

for some $\lambda > 0$, and we fix $A > a$. By [17, Theorem 1] there exists a function γ defined on (a, A) such that

$$w(r) < \gamma(r) \quad \forall r \in (a, A)$$

for any solution of (3.3.1) on (a, A). Moreover, γ can be assumed convex, and $\lim\limits_{t\to a}\gamma(r) = \lim\limits_{r\to A}\gamma(r) = +\infty$. If w is a solution of (3.3.1) on $(a, a+\epsilon)$ such that $w(a) > \min\limits_{a<r<A}\gamma(r)$ and $w'(a) > 0$, it is clear that $w(r^*) = \gamma(r^*)$ for some $r^* < A$ and $w(r) > \gamma(r)$ for $r \in (r^*, r^* + \epsilon)$, so w can not be defined on the whole (a, A), and there exists $a^* < A$ such that $\lim\limits_{r\to a^*} w(r) = \infty$. □

Proof of Theorem 3.1.5 By the Picard-Lipschitz fixed point theorem in the case $1 < p < 2$ and [8, Th 5.2] in the case $p \geq 2$, there exists a unique solution w_a to (3.1.16) defined on a maximal interval $[0, r_a)$ and w_a is an increasing function. Since Keller-Osserman estimate does not hold, by Lemma 3.3.1, the solution can be continued on the whole $[0, +\infty)$ and global uniqueness follows from the local uniqueness. The function $r \mapsto w_a(r)$ is increasing and

$$w_a(r) \geq a + \frac{p-1}{p}\Big(\frac{f(a)}{N}\Big)^{\frac{1}{p-1}} r^{\frac{p}{p-1}} \quad \text{and} \quad w_a'(r) \geq \Big(\frac{f(a)}{N}\Big)^{\frac{1}{p-1}} r^{\frac{1}{p-1}}$$

for any $r > 0$. □

Proposition 3.3.2 *Assume $p > \frac{2N}{N+1}$, f is locally Lipschitz continuous and $K = \infty$. For any positive function $u_0 \in C(Q_\infty)$ which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \tag{3.3.4}$$

for some $0 < a < b$, there exists a positive function $\overline{u} \in C(\overline{Q_\infty})$ solution of (3.1.1) *in Q_∞ and satisfying $\overline{u}(.,0) = u_0$ in $\mathbb{R}^N$. Furthermore*

$$w_a(|x|) \leq \overline{u}(x,t) \leq w_b(|x|) \quad \forall (x,t) \in Q_\infty. \tag{3.3.5}$$

Proof. Clearly w_a and w_b are ordered solutions of (3.1.1). We denote by u_n the solution to the initial-boundary problem

$$\left\{\begin{array}{rll} \partial_t u_n - \Delta_p u_n + f(u_n) & = 0 & \text{in } Q_n := B_n \times (0,\infty) \\ u_n(x,0) & = u_0(x) & \text{in } B_n \\ u_n(x,t) & = (w_a(|x|) + w_b(|x|))/2 & \text{on } \partial B_n \times (0,\infty). \end{array}\right. \tag{3.3.6}$$

By the maximum principle, u_n satisfies (3.3.5) in Q_n. Using locally parabolic equation regularity [5, Th 1.1, chap III] if $p \geq 2$ or [5, Th 1.1, chap IV] if $1 < p < 2$, we derive that the set of functions $\{u_n\}$ is eventually equicontinuous on any compact subset of $\overline{Q_\infty}$. Using a diagonal sequence, combined with Proposition 3.4.4, we conclude that there exists a subsequence $\{u_{n_k}\}$ which converges locally uniformly in $\overline{Q_\infty}$ to some solution $\overline{u} \in C(\overline{Q_\infty})$ which has the desired properties. □

Proposition 3.3.3 *Assume $p > \frac{2N}{N+1}$, f is locally Lipschitz continuous and $J = \infty$ and $K = \infty$. Then for any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$0 \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \tag{3.3.7}$$

for some $0 < b$, there exists a positive solution $\underline{u} \in C(\overline{Q_\infty})$ of (3.1.1) *in Q_∞ satisfying $\underline{u}(.,0) = u_0$ in $\mathbb{R}^N$ and*

$$\underline{u}(x,t) \leq \min\{w_b(|x|), \phi_\infty(t)\} \quad \forall (x,t) \in Q_\infty. \tag{3.3.8}$$

Proof. For any $R > 0$, let u_R be the solution of

$$\begin{cases} \partial_t u_R - \Delta_p u_R + f(u_R) & = 0 & \text{in } Q_\infty \\ u_R(x,0) & = u_0(x)\chi_{B_R}(x) & \text{in } \mathbb{R}^N. \end{cases}$$

The functions ϕ_∞ and w_b are solutions of (3.1.1) in Q_∞, which dominate u_R at $t = 0$, therefore, by the maximum principle,

$$\min\{\phi_\infty(t), w_b(|x|)\} \geq u_R(x,t) \quad \forall (x,t) \in Q_\infty. \tag{3.3.9}$$

The mapping $R \mapsto u_R$ is increasing, jointly with (3.3.9) it implies that there exists a solution $\underline{u} := \lim_{R\to\infty} u_R$ of (3.1.1) in Q_∞ which satisfies $\underline{u}(x,0) = u_0(x)$ in $\mathbb{R}^N$. Letting $R \to \infty$ in (3.3.9) yields to (3.3.8). □

Proof of Theorem 3.1.6. Combining Proposition 3.3.2 and Proposition 3.3.3 we see that there exist two solutions $\underline{u}$ and $\overline{u}$ with the same initial data u_0, which are ordered and different since $\lim_{|x|\to\infty} \overline{u}(x,t) = \infty$ and $\lim_{|x|\to\infty} \underline{u}(x,t) \leq \phi_\infty(t) < \infty$ for all $t > 0$. □

3.4 Estimates and stability

In this section we assume that Ω is a domain in $\mathbb{R}^N$, possibly unbounded, $0 < T \leq \infty$ and set $Q_T^\Omega := \Omega \times (0,T)$ and $Q_T := \mathbb{R}^N \times (0,T)$. We denote by $\mathfrak{M}(\Omega)$ the set of Radon measures in Ω and by $\mathfrak{M}_+(\Omega)$ its positive cone.

Definition 3.4.1 *A nonnegative function u is called a weak solution of* (3.1.1) *in Q_T^Ω if u, $|\nabla u|^p$, $f(u) \in L^1_{loc}(Q_T^\Omega)$ and*

$$\int_0^T \int_\Omega \Big(-G(u)\partial_t\varphi + |\nabla u|^{p-2}\,\nabla u.\nabla(g(u)\varphi) + f(u)g(u)\varphi\Big)\,dxdt = 0 \tag{3.4.1}$$

for any $\varphi \in C_c^\infty(Q_T^\Omega)$ and any function $g \in C(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ where $G'(r) = g(r)$.

The next results are obtained by adapting the proofs in [2].

3.4.1 Regularity properties

The following integral estimates are essentially [2, Prop 2.1] with u^q replaced by $f(u)$.

Proposition 3.4.2 *Assume $p > 1$. Let $\delta < 0$, $\delta \neq -1$ and $0 < t < \theta < T$. Let u be a nonnegative weak solution of* (3.1.1) *in Q_T^Ω. For any nonnegative function $\zeta \in C_c^\infty(\Omega)$ and $\tau > p$,*

$$\begin{aligned}\frac{1}{\delta+1}\int_\Omega (1+u(x,t))^{1+\delta}\zeta^\tau(x)dx + \frac{|\delta|}{2}\int_t^\theta\int_\Omega (1+u)^{\delta-1}\zeta^\tau |\nabla u|^p \,dx\,dt \\ \leq \frac{1}{\delta+1}\int_\Omega (1+u(x,\theta))^{1+\delta}\zeta^\tau(x)dx + \int_t^\theta\int_\Omega (1+u)^\delta f(u)\zeta^\tau dx\,dt \\ + c_{14}\int_t^\theta\int_\Omega (1+u)^{\delta+p-1}\zeta^{\tau-p}|\nabla\zeta|^p \,dx\,dt.\end{aligned} \tag{3.4.2}$$

and

$$\begin{aligned}\int_\Omega (1+u(x,t))\zeta^\tau(x)dx \leq \int_\Omega (1+u(x,\theta))\zeta^\tau(x)dx + \int_t^\theta\int_\Omega f(u)\zeta^\tau dx\,dt \\ + \tau\int_t^\theta\int_\Omega (1+u)^{\delta-1}\zeta^\tau |\nabla u|^p \,dx\,dt + \tau\int_t^\theta\int_\Omega (1+u)^{(1-\delta)(p-1)}\zeta^{\tau-p}|\nabla\zeta|^p \,dx\,dt.\end{aligned} \tag{3.4.3}$$

Conversely,

$$\begin{aligned}\frac{1}{4}\int_\Omega u(x,\theta)\zeta^\tau(x)dx + \frac{1}{2}\int_t^\theta\int_\Omega f(u)\zeta^\tau dx\,dt \\ \leq \int_\Omega u(x,t)\zeta^\tau(x)dx + \tau\int_t^\theta\int_\Omega \zeta^{\tau-1}|\nabla u|^{p-1}|\nabla\zeta| \,dx\,dt + c_{15}\end{aligned} \tag{3.4.4}$$

and

$$\begin{aligned}\frac{1}{4}\int_\Omega (1+u(x,\theta))\zeta^\tau(x)dx + \frac{1}{2}\int_t^\theta\int_\Omega f(u)\zeta^\tau dx\,dt \leq \int_\Omega (1+u(x,t))\zeta^\tau(x)dx \\ + \tau\int_t^\theta\int_\Omega (1+u)^{\delta-1}\zeta^\tau |\nabla u|^p \,dx\,dt + \tau\int_t^\theta\int_\Omega (1+u)^{(1-\delta)(p-1)}\zeta^{\tau-p}|\nabla\zeta|^p \,dx\,dt + c_{16}\end{aligned} \tag{3.4.5}$$

where $c_i = c_i(p, f, \delta, \tau)$ $(i = 18, 19, 20)$.

The next result is the keystone for the existence of an initial trace in the class of Radon measures. It is essentially [2, Prop 2.2] with u^q replaced by $f(u)$, but we shall sketch its proof for the sake of completeness.

Proposition 3.4.3 *Let u be a nonnegative weak solution of* (3.1.1) *in Q_T^Ω. Let $0 < \theta < T$. Assume that two of the three following conditions hold, for any open set $U \subset\subset \Omega$:*

$$\sup_{t\in(0,\theta]}\int_U u(x,t)dx < \infty, \tag{3.4.6}$$

$$\int_0^\theta \int_U f(u)dx\,dt < \infty, \tag{3.4.7}$$

$$\int_0^\theta \int_U |\nabla u|^{p-1}\,dx\,dt < \infty. \tag{3.4.8}$$

Then the third one holds for any $U \subset\subset \Omega$. Moreover,

$$\int_0^\theta \int_U u^\sigma dx\,dt < \infty \quad \forall \sigma \in (0, q_c) \tag{3.4.9}$$

and

$$\int_0^\theta \int_U |\nabla u|^r\,dx\,dt < \infty \quad \forall r \in (0, \frac{N}{N+1}q_c) \tag{3.4.10}$$

where $q_c = p - 1 + p/N$. Finally, there exists a Radon measure $\mu \in \mathfrak{M}_+(\Omega)$ such that for any $\zeta \in C_c(\Omega)$,

$$\lim_{t\to 0} \int_\Omega u(x,t)\zeta(x)dx = \int_\Omega \zeta(x)d\mu \tag{3.4.11}$$

and u satisfies

$$\begin{aligned}\int_0^\theta \int_\Omega (-u\partial_t\varphi + |\nabla u|^{p-2}\,\nabla u.\nabla\varphi + f(u)\varphi)dx\,dt \\ = \int_\Omega \varphi(x,0)d\mu - \int_\Omega u(x,\theta)\varphi(x,\theta)dx\end{aligned} \tag{3.4.12}$$

for any $0 < \theta < T$ and $\varphi \in C_c^\infty(\Omega \times [0,T))$.

Proof. Step 1 : Assume (3.4.6) *and* (3.4.8) *hold.* Let ζ and τ as in Proposition 3.4.2, there holds

$$\begin{aligned}\int_\Omega (1+u(x,t))\zeta^\tau dx = \int_\Omega (1+u(x,\theta))\zeta^\tau dx + \int_t^\theta \int_\Omega f(u)\zeta^\tau dxdt \\ + \tau\int_t^\theta \int_\Omega \zeta^{\tau-1}|\nabla u|^{p-2}\nabla u.\nabla\zeta dxdt.\end{aligned} \tag{3.4.13}$$

It follows that $f(u) \in L^1((0,\theta), L^1_{loc}(\Omega))$.

Step 2 : Assume that (3.4.7) *and* (3.4.8) *hold.* Then (3.4.6) follows from (3.4.13).

Step 3 : Assume that (3.4.6) *and* (3.4.7) *hold.* Let $\delta \in (\max(1-p,-1), 0)$ be fixed. From (3.4.2), we get, for any $0 < t < \theta$,

$$\begin{aligned}\frac{|\delta|}{2}\int_t^\theta \int_\Omega (1+u)^{\delta-1}\,|\nabla u|^p\,\zeta^\tau dx\,dt \leq \frac{1}{\delta+1}\int_\Omega (1+u(x,\theta))^{\delta+1}\zeta^\tau dx \\ + \int_t^\theta \int_\Omega (1+u)^\delta f(u)\zeta^\tau dx\,dt + c_{14}\int_t^\theta \int_\Omega (1+u)^{\delta+p-1}\zeta^{\tau-p}\,|\nabla\zeta|^p\,dx\,dt.\end{aligned} \tag{3.4.14}$$

If $p \leq 2$, then $(1+u)^{\delta+p-1} \leq 1+u$. Consequently, by (3.4.6),

$$\int_0^\theta \int_\Omega (1+u)^{\delta+p-1}\zeta^{\tau-p}\,|\nabla\zeta|^p\,dx\,dt < \int_0^\theta \int_\Omega (1+u)\zeta^{\tau-p}\,|\nabla\zeta|^p\,dx\,dt < \infty, \tag{3.4.15}$$

which, along with (3.4.7) and (3.4.14), implies that

$$\int_t^\theta \int_\Omega (1+u)^{\delta-1} |\nabla u|^p \zeta^\tau dx\, dt < c_{17}. \tag{3.4.16}$$

If $p > 2$, we choose $\delta \in (1-p, 2-p)$, $\delta \neq -1$, ζ and τ as in Proposition 3.4.2, then (3.4.2) remains valid. From the inequality $(1+u)^{1+\delta} < 1+u$ and (3.4.6), we find that

$$\frac{1}{|\delta+1|}\int_\Omega (1+u(x,t))^{1+\delta}\zeta^\tau(x)dx < \frac{1}{|\delta+1|}\int_\Omega (1+u(x,t))\zeta^\tau(x)dx < c_{18}.$$

Hence, by (3.4.2),

$$\begin{aligned}\frac{|\delta|}{2}\int_t^\theta \int_\Omega (1+u)^{\delta-1}\zeta^\tau |\nabla u|^p\, dx\, dt &\leq \frac{1}{\delta+1}\int_\Omega (1+u(x,\theta))^{\delta+1}\zeta^\tau dx \\ &+ \int_t^\theta \int_\Omega (1+u)^\delta f(u)\zeta^\tau dx\, dt + c_{14}\int_t^\theta \int_\Omega (1+u)^{\delta+p-1}\zeta^{\tau-p} |\nabla \zeta|^p\, dx\, dt + c_{18}.\end{aligned} \tag{3.4.17}$$

Since $\delta < 2-p$, $\delta+p-1 < 1$, hence $(1+u)^{\delta+p-1} \leq 1+u$. Therefore, (3.4.16) follows from (3.4.6), (3.4.7) and (3.4.17).

By applying the Gagliardo-Nirenberg inequality as in [2, Prop 2.2 (iii)], we deduce that

$$\int_0^\theta \int_U (1+u(x,t))^\sigma dx < c_{19}$$

for any $\sigma \in (0, q_c)$ with $q_c = p-1+\frac{p}{N}$, which leads to (3.4.9). Next for $0 < r < p$, and any $\delta < 0$, we find

$$\begin{aligned}\int_0^\theta \int_U |\nabla u|^r\, dx &\leq \left(\int_0^\theta \int_U (1+u)^{\delta-1} |\nabla u|^p\, dx\, dt\right)^{\frac{r}{p}} \\ &\times \left(\int_0^\theta \int_U (1+u)^{\frac{(1-\delta)r}{p-r}} dx\, dt\right)^{\frac{p-r}{p}}.\end{aligned} \tag{3.4.18}$$

Thus, if $r \in (0, \frac{Nq_c}{N+1})$, this proves (3.4.10) ; furthermore, since $p-1 < \frac{Nq_c}{N+1}$, we obtain (3.4.8).

Step 4 : End of the proof. Now we use (3.4.1) with $g = 1$, for any $\zeta \in C_c^\infty(\Omega)$ and any $0 < t < \theta < T$,

$$\int_\Omega u(x,t)\zeta(x)dx = \int_\Omega u(x,\theta)\zeta(x)dx + \int_t^\theta \int_\Omega \left(|\nabla u|^{p-2} \nabla u.\nabla\zeta + f(u)\zeta\right) dx\, dt. \tag{3.4.19}$$

Because the right-hand side of (3.4.19) has a finite limit when $t \to 0$, the same holds with $t \mapsto \displaystyle\int_\Omega u(x,t)\zeta(x)dx$. The mapping $\zeta \mapsto \lim_{t\to 0}\displaystyle\int_\Omega u(x,t)\zeta(x)dx$ is a positive linear functional ℓ_Ω on the space $C_c^\infty(\Omega)$. By a partition of unity it can be extended in a unique way as a Radon measure $\mu \in \mathfrak{M}_+(\Omega)$ and (3.4.11) holds.

Finally, let $0 < t < \theta$ be fixed, $g = 1$ and $\varphi \in C_c^\infty(Q_T^\Omega)$, thus

$$\begin{aligned}\int_t^\theta \int_\Omega (-u\partial_t\varphi + |\nabla u|^{p-2}\nabla u.\nabla\varphi + f(u)\varphi)dx\,d\tau \\ = \int_\Omega u(x,t)\varphi(x,0)dx - \int_\Omega u(x,\theta)\varphi(x,\theta)dx.\end{aligned} \tag{3.4.20}$$

But

$$\left|\int_\Omega u(x,t)(\varphi(x,t)-\varphi(x,0))dx\right| \leq c_{20}t\int_\Omega u(x,t)dx.$$

By (3.4.11), letting $t \to 0$ yields

$$\int_\Omega u(x,t)\varphi(x,t)dx \to \int_\Omega \varphi(x,0)d\mu.$$

Thus, letting $t \to 0$ in (3.4.20) implies (3.4.12). □

Next we consider the the following problems

$$\begin{cases} \partial_t u - \Delta_p u + f(u) & = 0 \quad \text{in } Q_T^\Omega, \\ u & = 0 \quad \text{on } \partial\Omega \times (0,T) \\ u(.,0) & = \mu \quad \text{in } \Omega. \end{cases} \tag{3.4.21}$$

where $\mu \in \mathfrak{M}_+(\Omega)$. The solutions are considered in the entropy sense (see [16] and [13]).

We recall that for $q \geq 1$ and $\Theta \subset \mathbb{R}^d$ open, the Marcinkiewicz space (or weak Lebesgue space) $M^q(\Theta)$ is the set of all locally integrable functions $u : \Theta \to \mathbb{R}$ such that there exists $C \geq 0$ with the property that for any measurable set $E \subset \Theta$,

$$\int_E |u|dy \leq C|E|^{1-\frac{1}{q}}. \tag{3.4.22}$$

The norm of u in $M^q(\Theta)$ is the smallest constant such that (3.4.22) holds for any measurable set E (see [16], [13] for more details). Here dy denotes the Lebesgue measure in $\mathbb{R}^d$, although any positive Borel measure can be used.

We recall the following result of Segura de Leon and Toledo [16, Th 2] and Li [13, Th 1.1] dealing with entropy solutions with initial data in L^1. However such solutions coincide with the semi-group solutions because of uniqueness.

Proposition 3.4.4 *Assume $p > \frac{2N}{N+1}$, $\Omega \subset \mathbb{R}^N$ is any open subset, $h \in L^1(Q_T^\Omega)$ and $\mu \in L^1_+(\Omega)$. Let $v \in C([0,T;L^1(\Omega))$ be the entropy solution to problem*

$$\begin{cases} \partial_t v - \Delta_p v & = h \quad \text{in } Q_T^\Omega \\ v & = 0 \quad \text{on } \partial\Omega \times (0,\infty) \\ v(.,0) & = \mu \quad \text{in } \Omega. \end{cases} \tag{3.4.23}$$

Then $v \in M^{p-1+\frac{p}{N}}(Q_T^\Omega)$, $|\nabla v| \in M^{p-\frac{N}{N+1}}(Q_T^\Omega)$ and there holds

$$\|v\|_{M^{p-1+\frac{p}{N}}(Q_T^\Omega)} + \|\nabla v\|_{M^{p-\frac{N}{N+1}}(Q_T^\Omega)} \leq c_{21}, \tag{3.4.24}$$

for some $c_{21} > 0$ depending on p, N, $\|\mu\|_{L^1(\Omega)}$ and $\|h\|_{L^1(Q_T^\Omega)}$.

3.4.2 Stability

Let $\{\mu_n\} \subset L^1_+(\mathbb{R}^N)$ be a sequence converging to μ in weak sense of measures, then $\|\mu_n\|_{L^1(\mathbb{R}^N)} \leq c^*$, where c^* depends only on N, p and $\|\mu\|_{\mathfrak{M}(\mathbb{R}^N)}$. Denote by u_{μ_n} (resp. v_{μ_n}) the solution to problem (3.4.21) (resp. (3.4.23) with $h \equiv 0$) with the initial data μ_n. Then the following estimate holds

$$0 \leq u_{\mu_n} \leq v_{\mu_n}. \tag{3.4.25}$$

By [9, Theorem 3],

$$\|v_{\mu_n}(.,t)\|_{L^\infty(\mathbb{R}^N)} \leq c_{22} t^{\frac{-N}{N(p-2)+p}} \|\mu_n\|_{L^1(\mathbb{R}^N)}^{\frac{p}{N(p-2)+p}} \quad \forall t > 0,$$

where $c_{22} = c_{22}(N,p) > 0$. Thus

$$\begin{aligned} \|u_{\mu_n}(.,t)\|_{L^\infty(\mathbb{R}^N)} &\leq c_{22} t^{\frac{-N}{N(p-2)+p}} \|\mu_n\|_{L^1(\mathbb{R}^N)}^{\frac{p}{N(p-2)+p}} \\ &\leq c_{23} t^{\frac{-N}{N(p-2)+p}} \end{aligned} \tag{3.4.26}$$

for every $t > 0$, where $c_{23} = c_{23}(N,p,c^*) > 0$.

It follows from (3.4.24) and (3.4.25) that

$$\|u_{\mu_n}\|_{M^{p-1+p/N}(Q_T)} \leq c_{21} \|\mu_n\|_{L^1(\mathbb{R}^N)}^{\frac{p+N}{1+p(N-1)}} \leq c_{24}(N,p,c^*). \tag{3.4.27}$$

By (3.4.26) and the regularity theory of degenerate parabolic equations [5], we derive that the sequence $\{u_{\mu_n}\}$ is equicontinuous in any compact subset of Q_T. As a consequence, there exist a subsequence, still denoted by $\{u_{\mu_n}\}$ and a function u such that $\{u_{\mu_n}\}$ converges to u locally uniformly in Q_T.

Lemma 3.4.5 *The sequence $\{f(u_{\mu_n})\}$ converges strongly to $f(u)$ in $L^1(Q_T)$. Furthermore, $\{u_n\}$ converges strongly to u in $L^q_{loc}(\overline{Q_T})$ for every $1 \leq q < q_c$.*

Proof. Since $u_\mu \to u$ a.e in Q_T, by Vitali's theorem, it is sufficient to show that the sequence $\{f(u_{\mu_n})\}$ is uniformly integrable. Let E be a Borel subset of Q_T and let $R > 0$. Then, since f is increasing,

$$\begin{aligned} \int\!\!\int_E f(u_{\mu_n}) dx\, dt &= \int\!\!\int_{E \cap \{u_{\mu_n} \leq R\}} f(u_n) dx\, dt + \int\!\!\int_{E \cap \{u_{\mu_n} > R\}} f(u_{\mu_n}) dx\, dt \\ &\leq f(R) \int\!\!\int_E dx\, dt + \int\!\!\int_{E \cap \{u_{\mu_n} > R\}} f(u_{\mu_n}) dx\, dt. \end{aligned}$$

For $\lambda \geq 0$, we set $B_n(\lambda) = \{(x,t) \in Q_T) : u_{\mu_n} > \lambda\}$ and $a_n(\lambda) = \displaystyle\int\!\!\int_{B_n(\lambda)} dx\, dt$. Then

$$\int\!\!\int_{E \cap \{u_{\mu_n} > R\}} f(u_{\mu_n}) dx\, dt \leq \int\!\!\int_{\{u_{\mu_n} \geq R\}} f(u_{\mu_n}) dx\, dt = -\int_R^\infty f(\lambda) da_n(\lambda) \tag{3.4.28}$$

and

$$-\int_R^\infty f(\lambda) da_n(\lambda) \leq f(R) a_n(R) + \int_R^\infty a_n(\lambda) df(\lambda).$$

It follows from (3.4.27) that

$$a_n(\lambda) \le c_{21} \|\mu_n\|_{\mathfrak{M}_+(\mathbb{R}^N)}^{\frac{p+N}{1+p(N-1)}} \lambda^{-(p-1+\frac{p}{N})} \le c_{25}\lambda^{-(p-1+\frac{p}{N})}.$$

Plugging these estimates into (3.4.28) yields

$$\begin{aligned} \int\int_{E\cap\{u_{\mu_n}>R\}} f(u_{\mu_n})dx\,dt &\le f(R)a_n(R) + c_{25}\int_R^\infty \lambda^{-(p-1+\frac{p}{N})}df(\lambda) \\ &\le f(R)a_n(R) - c_{25}f(R)R^{-(p-1-\frac{p}{N})} \\ &\quad + c_{25}\left(p-1+\frac{p}{N}\right)\int_R^\infty f(\lambda)\lambda^{-(p+\frac{p}{N})}d\lambda \\ &\le c_{25}\left(p-1+\frac{p}{N}\right)\int_R^\infty f(\lambda)\lambda^{-(p+\frac{p}{N})}d\lambda. \end{aligned} \tag{3.4.29}$$

Since

$$\int_1^\infty \lambda^{-(p+\frac{p}{N})}f(\lambda)d\lambda < \infty,$$

for given $\epsilon > 0$,we can choose $R > 0$ large enough such that

$$c_{25}\left(p-1+\frac{p}{N}\right)\int_R^\infty f(\lambda)\lambda^{-(p+\frac{p}{N})}d\lambda < \frac{\epsilon}{2}.$$

Set $\delta = (1+f(R))^{-1}\frac{\epsilon}{2}$, then

$$|E| < \delta \Longrightarrow 0 \le \int\int_E f(u_{\mu_n})dx\,dt < \epsilon,$$

which proves the uniform integrability of the sequence $\{f(u_{\mu_n})\}$. The last assertion follows from the fact that u_{μ_n} is bounded in $M^{q_c}(Q_T)$ (remember that $q_c = p-1+\frac{p}{N}$) and $M^{q_c}(Q_T) \subset L^q_{loc}(\overline{Q_T})$ with continuous imbedding, for any $q < q_c$. The conclusion follows again by Vitali's theorem. ☐

Lemma 3.4.6 *Assume $p > \frac{2N}{N+1}$, then for any $U \subset\subset \mathbb{R}^N$, the sequence $\{\nabla u_{\mu_n}\}$ converges strongly to ∇u in $(L^s(Q_T))^N$ for every $1 \le s < s_c := p - \frac{N}{N+1}$.*

Proof. We set $h_n = -f(u_{\mu_n})$ and write the equation under the form

$$\left\{\begin{aligned} \partial_t u_{\mu_n} - \Delta_p u_{\mu_n} &= h_n \quad \text{in } Q_T \\ u_{\mu_n}(.,0) &= \mu_n \quad \text{in } \mathbb{R}^N. \end{aligned}\right. \tag{3.4.30}$$

We already know from the L^1-contraction principle and Proposition 3.4.4 that

$$\|u_{\mu_n}(.,t)\|_{L^1(\mathbb{R}^N)} \le \|\mu_n\|_{L^1(\mathbb{R}^N)} \qquad \forall t \in (0,T]$$

and $u_{\mu_n} \to u$ in $L^q_{loc}(\overline{Q_T})$ for every $q \in [1,q_c)$ and $|\nabla u_{\mu_n}|$ is bounded in $L^s_{loc}(\overline{Q_T})$ for every $1 \le s < s_c$. Thus $|\nabla u_{\mu_n}|^{p-1}$ remains bounded in bounded in $L^\sigma_{loc}(\overline{Q_T})$ for every $1 \le \sigma < \sigma_c := 1 + \frac{1}{(N+1)(p-1)}$. Furthermore,

$$\{\nabla u_{\mu_n}\} \text{ is a Cauchy sequence in measure.} \tag{3.4.31}$$

and the proof is similar to the one of [2, Th 5.1-step2]. Up to the extraction of a subsequence, $\{\nabla u_{\mu_n}\}$ converges a.e. to some $D = (D_1, ..., D_N)$ in Q_T. Consequently, $\{|\nabla u_{\mu_n}|^{p-2}\nabla u_{\mu_n}\}$ converges a.e. to $|D|^{p-2}D$ in Q_T and, by Vitali's theorem,

$$\begin{aligned} \nabla u_{\mu_n} \to D \qquad & \text{strongly in } (L^s_{loc}(\overline{Q_T}))^N, \quad \forall s \in [1, s_c), \\ \{|\nabla u_{\mu_n}|^{p-2}\nabla u_{\mu_n}\} \to |D|^{p-2}D \qquad & \text{strongly in } (L^\sigma_{loc}(\overline{Q_T}))^N, \quad \forall \sigma \in [1, \sigma_c). \end{aligned} \tag{3.4.32}$$

which implies $\nabla u = D$ and the conclusion of the lemma follows. □

Proof of Theorem 3.1.7. *Step 1.* For any $\zeta \in C^\infty_c(\mathbb{R}^N)$ and $t > 0$, we have

$$\int_{\mathbb{R}^N} u_{\mu_n}(x,t)\zeta(x)dx + \int_0^t \int_{\mathbb{R}^N} (|\nabla u_{\mu_n}|^{p-2}\nabla u_{\mu_n}\nabla\zeta + f(u_{\mu_n})\zeta)dx\,dt = \int_{\mathbb{R}^N} \mu_n(x)\zeta(x)dx.$$

By Lemma 3.4.5 and Lemma 3.4.6, up to the extraction of a subsequence, we can pass to the limit in each term and get

$$\int_{\mathbb{R}^N} u(x,t)\zeta(x)dx + \int_0^t \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u\nabla\zeta + f(u)\zeta)dx\,dt = \int_{\mathbb{R}^N} \zeta d\mu.$$

Letting $t \to 0$ yields

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N} \zeta(x). \tag{3.4.33}$$

For any $\varphi \in C^\infty_c(\mathbb{R}^N \times [0,\infty))$ and $\theta > 0$, we have

$$\begin{aligned} &\int_0^\theta \int_{\mathbb{R}^N} (-u_{\mu_n}\partial_t\varphi + |\nabla u_{\mu_n}|^{p-2}\nabla u_{\mu_n}.\nabla\varphi + f(u_{\mu_n})\varphi)dx\,dt \\ &\qquad = \int_{\mathbb{R}^N} \varphi(0,x)\mu_n(x)dx - \int_{\mathbb{R}^N} u_{\mu_n}(x,\theta)\varphi(x,\theta)dx. \end{aligned} \tag{3.4.34}$$

By the previous convergence results, we can pass to the limit in (3.4.34) to obtain

$$\begin{aligned} &\int_0^\theta \int_{\mathbb{R}^N} (-u\partial_t\varphi + |\nabla u|^{p-2}\nabla u\nabla\varphi + f(u)\varphi)dx\,dt \\ &\qquad\qquad = \int_{\mathbb{R}^N} \varphi(.,0)d\mu - \int_{\mathbb{R}^N} u(.,\theta)\varphi(.,\theta)dx. \end{aligned} \tag{3.4.35}$$

Step 2 : u is a weak solution. By (3.4.26)

$$\sup\{\|u_{\mu_n}(.,t)\|_{L^\infty(\mathbb{R}^N)}, \|u(.,t)\|_{L^\infty(\mathbb{R}^N)}\} \le c_{23}t^{-\frac{N}{N(p-2)+p}} \qquad \forall t \in (0,T].$$

Let $\zeta \in C^\infty_c(\mathbb{R}^N)$. Since $\{u_{\mu_n}(.,\theta)\}$ converges locally uniformly to $u(.,\theta)$ in $\mathbb{R}^N$, for any $\theta > 0$, there holds

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} (u_{\mu_n} - u_{\mu_m})^2(.,T)\zeta dx\,dt + \int_\theta^T \int_{\mathbb{R}^N} (f(u_{\mu_n}) - f(u_{\mu_m}))(u_{\mu_n} - u_{\mu_m})\zeta dx\,dt \\ &\quad + \int_\theta^T \int_{\mathbb{R}^N} (|\nabla u_{\mu_n}|^{p-2}\nabla u_{\mu_n} - |\nabla u_{\mu_m}|^{p-2}\nabla u_{\mu_m}).\nabla(u_{\mu_m} - u_{\mu_n})\zeta dx\,dt \\ &\quad \le \frac{1}{2}\int_{\mathbb{R}^N} (u_{\mu_n} - u_{\mu_m})^2(.,\theta)\zeta dx\,dt \\ &\quad + \int_\theta^T \int_{\mathbb{R}^N} \left||\nabla u_{\mu_n}|^{p-2}\nabla u_{\mu_n} - |\nabla u_{\mu_m}|^{p-2}\nabla u_{\mu_m}\right| |u_{\mu_m} - u_{\mu_n}| |\nabla\zeta| dx\,dt. \end{aligned} \tag{3.4.36}$$

This implies directly

$$\nabla u_{\mu_n} \to \nabla u \text{ in } (L^p_{loc}(Q_T))^N, \tag{3.4.37}$$

by Lemma 3.4.6 when $p \geq 2$. When $1 < p < 2$, we derive by Fatou's lemma

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N}(u_{\mu_n}-u)^2(.,T)\zeta dx\,dt + \int_\theta^T\int_{\mathbb{R}^N}(f(u_{\mu_n})-f(u))(u_{\mu_n}-u)\zeta dx\,dt \\ &\quad + \int_\theta^T\int_{\mathbb{R}^N}(|\nabla u_{\mu_n}|^{p-2}\nabla u_{\mu_n} - |\nabla u|^{p-2}\nabla u).\nabla(u_{\mu_n}-u)\zeta dx\,dt \\ &\quad \leq \frac{1}{2}\int_{\mathbb{R}^N}(u_{\mu_n}-u)^2(.,\theta)\zeta dx\,dt \\ &\quad + \int_\theta^T\int_{\mathbb{R}^N}\left||\nabla u_{\mu_n}|^{p-2}\nabla u_{\mu_n} - |\nabla u|^{p-2}\nabla u\right||u_{\mu_m}-u|\,|\nabla\zeta|\,dx\,dt. \end{aligned} \tag{3.4.38}$$

Using again Lemma 3.4.6, it implies

$$\lim_{n\to\infty}\int_\theta^T\int_{\mathbb{R}^N}|\nabla u_{\mu_n}|^p\,\zeta dx\,dt = \int_\theta^T\int_{\mathbb{R}^N}|\nabla u|^p\,\zeta dx\,dt. \tag{3.4.39}$$

Since $\nabla u_{\mu_n} \rightharpoonup \nabla u$ weakly in $L^p_{loc}(Q_T)$, it implies again that (3.4.37) holds true. At end, let $\varphi \in C_c^\infty(Q_T)$ and consider $0 < \theta < T$ and $U \subset\subset \mathbb{R}^N$ such that $\operatorname{supp}\varphi \subset (\theta,T)\times U$. Let $g \in C(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$ where $G'(r) = g(r)$. Multiplying the equation in (3.4.21) (with initial data $\mu = \mu_n$) by $g(u_{\mu_n})\varphi$, we obtain

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}^N}(-G(u_{\mu_n})\partial_t\varphi + |\nabla u_{\mu_n}|^p\,g'(u_{\mu_n})\varphi)dx\,dt \\ &\qquad + g(u_{\mu_n})\,|\nabla u_{\mu_n}|^{p-2}\,\nabla u_{\mu_n}.\nabla\varphi + \int_0^T\int_{\mathbb{R}^N}g(u_{\mu_n})f(u_{\mu_n})dx\,dt = 0. \end{aligned} \tag{3.4.40}$$

By Lemma 3.4.5 and (3.4.37), we can pass to the limit in each term. As a consequence, u is a weak solution.

Step 3 : Stability. Assume that $\{\mu_n\}$ is a sequence of functions in $L^1_+(\mathbb{R}^N)$ with compact support, which converges to $\mu \in \mathfrak{M}^b_+(\mathbb{R}^N)$ in the dual sense of $C(\mathbb{R}^N)$, then $\|\mu_n\|_{L^1(\mathbb{R}^N)}$ is bounded independently of n. By the same argument as in step 1 and step 2, we can pass to the limit in each term of (3.4.40), hence the conclusion follows. □

Lemma 3.4.7 *Assume $p \geq 2$. Let $u \in C(Q_T)$ be a positive weak solution of* (3.1.1) *in Q_T. Assume that there exists $r > 0$ such that*

$$\int_0^T\int_{B_r}|\nabla u|^{p-1}\,dx\,dt = \infty. \tag{3.4.41}$$

Then

$$\sup_{\tau\in(0,T)}\int_{B_{8r}}u(x,\tau) = \infty. \tag{3.4.42}$$

Proof. By contradiction we assume that (3.4.42) does not hold. Then there exists $M_1 > 0$ such that

$$\sup_{\tau\in(0,T)} \int_{B_{8r}} u(x,\tau) = M_1. \tag{3.4.43}$$

Step 1 : We claim that

$$u \in L^\infty(Q_T^{B_{2r}}).$$

Since u is a positive subsolution of the equation in (3.2.13), by [5, Theorem 4.2, Chapter V], there exists a constant $c_{26} = c_{26}(N,p)$ such that for every $x_0 \in \mathbb{R}^N$, $0 < \theta \le t_0 < T$ and $\sigma \in (0,1)$, there holds

$$\sup_{K_{\sigma\rho}(x_0)\times(t_0-\sigma\theta,t_0)} u \le \frac{c_{26}\theta^{\frac{1}{2}}}{\rho^{\frac{p}{2}}(1-\sigma)^{\frac{N(p+1)+p}{2}}} \left(\sup_{t_0-\theta<\tau<t_0} |K_\rho(x_0)|^{-1} \int_{K_\rho(x_0)} u(x,\tau)dx \right)^{\frac{p}{2}}, \tag{3.4.44}$$

where $K_\rho(x_0)$ is the cube centered at x_0 and wedge 2ρ, i.e.,

$$K_\rho(x_0) = \{x \in \mathbb{R}^N : \max_{1\le i\le N} \left|x^i - x_0^i\right| < \rho\}.$$

We choose $x_0 = 0$, $t_0 = \theta = t$, $\sigma = 1/2$ and $\rho = 4r$, then (3.4.44) becomes

$$\sup_{K_{2r}\times(\frac{t}{2},t)} u \le 2^{\frac{N(p+1)+p}{2}} c_{26} t^{\frac{1}{2}} (4r)^{\frac{-p}{2}} \left(\sup_{0<\tau<t} |K_{4r}|^{-1} \int_{K_{4r}} u(x,\tau)dx \right)^{\frac{p}{2}}. \tag{3.4.45}$$

Since $B_{2r} \subset K_{2r}$ and $K_{4r} \subset B_{8r}$, from (3.4.43) and (3.4.45), we obtain that

$$\sup_{B_{2r}\times(0,T)} u \le 2^{\frac{N-p(2N+1)}{2}} c_{26} T^{\frac{1}{2}} r^{\frac{-p(N+1)}{2}} M_1^{\frac{p}{2}} =: M_2. \tag{3.4.46}$$

If $p = 2$, (3.4.46) can be derived by using [5, remark 4.1, Chapter V].

Step 2 : Let $\zeta \in C_c^\infty(\mathbb{R}^N)$ such that $\zeta \ge 0$ in $\mathbb{R}^N$, $\zeta = 1$ in B_r, $\zeta = 0$ outside of B_{2r} and $|\nabla\zeta| \le \frac{1}{r}$. We show that, for any $\sigma > 0$,

$$\begin{aligned} J_1(t) &:= \int_0^t \int_{B_{2r}} (u+1)^{\sigma-1} \zeta^p \left|\nabla u\right|^p dx\, d\tau < \infty, \\ J_2(t) &:= \int_0^t \int_{B_{2r}} (u+1)^{(1-\sigma)(p-1)} dx\, d\tau < \infty. \end{aligned} \tag{3.4.47}$$

Multiplying (3.1.1) by $(u+1)^\sigma\zeta^p$ and then integrating on $\mathbb{R}^N \times [\epsilon, t]$ with $0 < \epsilon < t$, we get

$$\begin{aligned} \frac{1}{\sigma+1} \int_{B_{2r}} (u(x,t)+1)^{\sigma+1}\zeta^p dx + \sigma \int_\epsilon^t \int_{B_{2r}} (u+1)^{\sigma-1}\zeta^p \left|\nabla u\right|^p dx\, d\tau \\ + \int_\epsilon^t \int_{B_{2r}} (u+1)^\sigma f(u)\zeta^p dx\, d\tau \\ = \frac{1}{\sigma+1} \int_{B_{2r}} (u(x,\epsilon)+1)^{\sigma+1}\zeta^p dx \\ - p \int_\epsilon^t \int_{B_{2r}} (u+1)^\sigma \zeta^{p-1} \left|\nabla u\right|^{p-2} \nabla u \nabla\zeta dx\, d\tau, \end{aligned}$$

which implies that

$$\begin{aligned}\sigma&\int_\epsilon^t\int_{B_{2r}}(u+1)^{\sigma-1}\zeta^p\,|\nabla u|^p\,dx\,d\tau\\&\le\frac{1}{\sigma+1}\int_{B_{2r}}(u(x,\epsilon)+1)^{\sigma+1}\zeta^p dx+p\int_\epsilon^t\int_{B_{2r}}(u+1)^\sigma\zeta^{p-1}\,|\nabla u|^{p-1}\,|\nabla\zeta|\,dx\,d\tau.\end{aligned}\tag{3.4.48}$$

By Young's inequality, for any $\delta>0$,

$$\begin{aligned}p&\int_\epsilon^t\int_{B_{2r}}(u+1)^\sigma\zeta^{p-1}\,|\nabla u|^{p-1}\,|\nabla\zeta|\,dx\,dt\tau\\&\le p\delta\int_\epsilon^t\int_{B_{2r}}(u+1)^{\sigma-1}\zeta^p\,|\nabla u|^p\,dx\,d\tau+p\delta^{-\frac{1}{p-1}}\int_\epsilon^t\int_{B_{2r}}(u+1)^{\sigma+p-1}\,|\nabla\zeta|^p\,dx\,d\tau.\end{aligned}\tag{3.4.49}$$

It follows from (3.4.48) and (3.4.49) that

$$\begin{aligned}(\sigma-p\delta)&\int_\epsilon^t\int_{B_{2r}}(u+1)^{\sigma-1}\zeta^p\,|\nabla u|^p\,dx\,d\tau\\&\le\frac{1}{\sigma+1}\int_{B_{2r}}(u(x,\epsilon)+1)^{\sigma+1}\zeta^p dx+p\delta^{-\frac{1}{p-1}}\int_\epsilon^t\int_{B_{2r}}(u+1)^{\sigma+p-1}\,|\nabla\zeta|^p\,dx\,d\tau.\end{aligned}\tag{3.4.50}$$

By (3.4.46),

$$\sup_{\epsilon\in(0,T)}\int_{B_{2r}}(u(x,\epsilon)+1)^{\sigma+1}\zeta^p dx\le c_{27}$$

where $c_{27}=c_{27}(N,p,r,\zeta,M_2)$ and

$$\int_0^t\int_{B_{2r}}(u+1)^{\sigma+p-1}\,|\nabla\zeta|^p\,dx\,d\tau\le r^{-p}\int_0^t\int_{B_{2r}}(u+1)^{\sigma+p-1}dx\,dt\le c_{28}$$

where $c_{28}=c_{28}(N,p,r,T,M_2)$. By combining the previous two estimates with (3.4.50) and by choosing $\delta=\frac{\sigma}{2p}$, we get

$$J_1(t)\le c_{29}\quad\forall t\in(0,T)\tag{3.4.51}$$

where $c_{29}=c_{29}(N,p,r,T,\zeta)$. By (3.4.46), we also find that

$$J_2(t)\le c_{30}\tag{3.4.52}$$

where $c_{30}=c_{30}(N,p,r,T,M_2)$.

Step 3 : End of proof. By Hölder's inequality, we get

$$\int_0^t\int_{B_{2r}}|\nabla u|^{p-1}\,\zeta^{p-1}dx\,d\tau\le(J_1(t))^{\frac{p-1}{p}}(J_2(t))^{\frac{1}{p}}.$$

By step 2, we deduce that

$$\int_0^T\int_{B_{2r}}|\nabla u|^{p-1}\,\zeta^{p-1}dx\,dt<c_{31},\tag{3.4.53}$$

where $c_{31}=c_{31}(N,p,r,T,\zeta)$ which contradicts (3.4.41). □

3.5 Initial trace

From Proposition 3.4.3 and Lemma 3.4.7, we derive the dichotomy result Theorem 3.1.8.

Proof of Theorem 3.1.8 By translation we may suppose that $y = 0$.

Case 1 : there exists an open neighborhood U of 0 such that (3.4.7) and (3.4.8) hold true. Then the statement (ii) follows from Proposition 3.4.3.

Case 2 : for any open neighborhood U of 0, (3.4.7) or (3.4.8) does not hold. We first suppose that (3.4.8) does not hold. We can choose $r > 0$ such that $B_{8r} \subset U$ and (3.4.41) holds. Then the statement (i) follows from Lemma 3.4.7. Suppose next that (3.4.8) holds but (3.4.7) does not hold, then Proposition 3.4.3 implies that (3.4.6) does not hold and the statement (i) follows. □

Proposition 3.5.1 *Assume $p \geq 2$ and f satisfies (3.1.12). Let u is a positive weak solution of (3.1.1) in Q_∞ with initial trace $(\mathcal{S}, \mu)$. Then for every $y \in \mathcal{S}$,*

$$\underline{U}_y(x,t) := \underline{U}(x-y,t) \leq u(x,t) \tag{3.5.1}$$

in Q_∞.

Proof. By translation we may suppose that $y = 0$. Since $0 \in \mathcal{S}(u)$, for any $\eta > 0$ small enough

$$\lim_{t \to 0} \int_{B_\eta} u(x,t)dx = \infty.$$

For $\epsilon > 0$, denote $M_{\epsilon,\eta} = \displaystyle\int_{B_\eta} u(x,\epsilon)dx$. For any $m > m_\eta = \inf_{\sigma>0} M_{\sigma,\eta}$ there exists $\epsilon = \epsilon(m,\eta)$ such that $m = M_{\epsilon,\eta}$ and $\lim_{\eta \to 0} \epsilon(m,\eta) = 0$. Let $\tilde{u}_\eta$ be the solution to the problem

$$\begin{cases} \partial_t \tilde{u}_\eta - \Delta_p \tilde{u}_\eta + f(\tilde{u}_\eta) &= 0 \quad \text{in } Q_\infty \\ \tilde{u}_\eta(x,0) &= u(x,\epsilon)\chi_{B_\eta} \quad \text{in } \mathbb{R}^N \end{cases}$$

where χ_{B_η} is the characteristic function of B_η. By the maximum principle $\tilde{u}_\eta \leq u$ in $\mathbb{R}^N \times (\epsilon, \infty)$. By Theorem 3.1.7 $\tilde{u}_\eta$ converges to $u_{k\delta_0}$ when η goes to zero. Letting m go to infinity yields (3.5.1). □

Proof of Theorem 3.1.2 The conclusion follows directly from Proposition 3.5.1. □

Lemma 3.5.2 *Assume $p \geq 2$, (3.1.12) is satisfied, $J < \infty$ and $\lim_{k \to \infty} u_{k\delta_0}(x,t) = \phi_\infty(t)$ for every $(x,t) \in Q_\infty$. If u is a positive solution of (3.1.1) in Q_∞ which satisfies*

$$\limsup_{t \to 0} \int_G u(x,t)dx = \infty, \tag{3.5.2}$$

for some bounded open subset $G \subset \mathbb{R}^N$, then $u(x,t) \geq \phi_\infty(t)$ for every $(x,t) \in Q_\infty$.

Proof. By assumption, there exists a sequence $\{t_n\}$ decreasing to 0 such that

$$\lim_{n\to\infty}\int_G u(x,t_n)dx = \infty. \tag{3.5.3}$$

If (3.5.2) holds, we can construct a decreasing sequence of open subsets $G_k \subset G$ such that $\overline{G_k} \subset G_{k-1}$, $\operatorname{diam}(G_k) = \epsilon_k \to 0$ when $k \to \infty$, and

$$\lim_{n\to\infty}\int_{G_k} u(x,t_n)dx = \infty \quad \forall k \in \mathbb{N}. \tag{3.5.4}$$

Furthermore there exists a unique $a \in \cap_k G_k$. We set

$$\int_{G_k} u(x,t_n)dx = M_{n,k}.$$

Since $\lim\limits_{n\to\infty} M_{n,k} = \infty$, we claim that for any $m > 0$ and any k, there exists $n = n(k) \in \mathbb{N}$ such that

$$\int_{G_k} u(x,t_{n(k)})dx \geq m. \tag{3.5.5}$$

By induction, we define $n(1)$ as the smallest integer n such that $M_{n,1} \geq m$. This is always possible. Then we define $n(2)$ as the smallest integer larger than $n(1)$ such that $M_{n,2} \geq m$. By induction, $n(k)$ is the smallest integer n larger than $n(k-1)$ such that $M_{n,k} \geq m$. Next, for any k, there exists $\ell = \ell(k)$ such that

$$\int_{G_k} \inf\{u(x,t_{n(k)});\ell\}dx = m \tag{3.5.6}$$

and we set

$$\hat{U}_k(x) = \inf\{u(x,t_{n(k)});\ell\}\chi_{G_k}(x).$$

Let $u := \hat{u}_k$ be the unique bounded solution of

$$\begin{cases} \partial_t u - \Delta_p u + f(u) &= 0 \quad \text{in } Q_\infty \\ u(.,0) &= \hat{U}_k \quad \text{in } \mathbb{R}^N. \end{cases} \tag{3.5.7}$$

Since $\hat{u}_k(x,0) \leq u(x,t_{n(k)})$, we derive

$$u(x,t+t_{n(k)}) \geq \hat{u}_k(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{3.5.8}$$

When $k \to \infty$, $\hat{U}_k \to m\delta_a$, thus $\hat{u}_k \to u_{m\delta_a}$ by Theorem 3.1.7. Therefore $u \geq u_{m\delta_a}$. Since m is arbitrary and $u_{m\delta_a} \to \phi_\infty$ when $m \to \infty$, it follows that $u \geq \phi_\infty$. □

When $J = \infty$, the following phenomenon occurs :

Lemma 3.5.3 *Assume $p \geq 2$,* (3.1.12) *is satisfied, $J = \infty$ and $\lim\limits_{k\to\infty} u_{k\delta_0} = \infty$ in Q_∞. There exists no positive solution u of* (3.1.1) *in Q_∞ which satisfies* (3.5.2) *for some bounded open subset $G \subset \mathbb{R}^N$.*

Proof. If we assume that such a u exists, we proceed as in the proof of the previous lemma. Since Theorem 3.1.7 holds, we derive that $u \geq u_{m\delta_a}$ for any m. Since $\lim_{m\to\infty} u_{m\delta_a}(x,t) = \infty$ for all $(x,t) \in Q_\infty$, we are led to a contradiction. □

Thanks to these results, we can characterize the initial trace of positive solutions of (3.1.1) when the Keller-Osserman condition does not hold.

Proof of Theorem 3.1.9. *(i)* If $\mathcal{S}(u) \neq \emptyset$, there exists $y \in \mathcal{S}(u)$ and an open neighborhood G of y such that (3.5.2) holds. By Lemma 3.5.2, $u \geq \phi_\infty$ and the initial trace of u is the Borel measure ν_∞. Otherwise, $\mathcal{R}(u) = \mathbb{R}^N$ and $Tr_{_{\mathbb{R}^N}}(u) \in \mathfrak{M}_+(\mathbb{R}^N)$.

(ii) By the same argument as above and because of Lemma 3.5.3, $\mathcal{S}(u) = \emptyset$. Therefore $\mathcal{R}(u) = \mathbb{R}^N$ and $Tr_{_{\mathbb{R}^N}}(u) \in \mathfrak{M}_+(\mathbb{R}^N)$. □

Consequently, if $K = \infty$, we obtain the folollowing result :

Corollary 3.5.4 *Assume $p \geq 2$. If f satisfies* (3.1.8), (3.1.12), *$J < \infty$ and $K = \infty$, there exist infinitely many different positive solutions u of* (3.1.1) *such that $tr_{\mathbb{R}^N}(u) = \nu_\infty$.*

Proof. Let $b > 0$ be fixed. Since f satisfies (3.1.8), $(x,t) \mapsto U(x,t) = w_b(x) + \phi_\infty(t)$ is a supersolution for (3.1.1). Let $V(x,t) = \max\{w_b(x), \phi_\infty(t)\}$ then V, $f(V)$ and $|\nabla V|^p$ are locally integrable in Q_T ; actually V is locally Lipschitz continuous. Let $\epsilon > 0$ and ρ_ϵ be a smooth approximation defined by

$$\rho_\epsilon(r) = \begin{cases} 0 & \text{if} \quad r < 0 \\ \frac{r^2}{2\epsilon} & \text{if} \quad 0 < r < \epsilon \\ r - \frac{\epsilon}{2} & \text{if} \quad r > < \epsilon \end{cases}$$

We set $V_\epsilon(x,t) = \phi_\infty(t) + \rho_\epsilon[w_b(x) - \phi_\infty(t)]$. Then

$$\begin{aligned} \partial_t V_\epsilon - \Delta_p V_\epsilon + f(V_\epsilon) &= \phi'_\infty\,(1 - \rho'_\epsilon[w_b - \phi_\infty]) - (\rho'_\epsilon[w_b - \phi_\infty])^{p-1}\,\Delta_p w_b \\ &\quad - (p-1)\,(\rho'_\epsilon[w_b - \phi_\infty])^{p-2}\,\rho''_\epsilon[w_b - \phi_\infty]|\nabla w_b|^p + f(V_\epsilon) \\ &\leq f(V_\epsilon) - (1 - \rho'_\epsilon[w_b - \phi_\infty])\,f(\phi_\infty) - (\rho'_\epsilon[w_b - \phi_\infty])^{p-1}\,f(w_b). \end{aligned}$$

If $\phi \in C_c^\infty(Q_T)$ is nonnegative, then

$$\iint_{Q_T} \left(-V_\epsilon \partial_t \phi + |\nabla V_\epsilon|^{p-2} \nabla V_\epsilon . \nabla \phi + f(V_\epsilon)\phi\right) dx\,dt \leq o(1).$$

Letting $\epsilon \to 0$ implies

$$\iint_{Q_T} \left(-V \partial_t \phi + |\nabla V|^{p-2} \nabla V . \nabla \phi + f(V)\phi\right) dx\,dt \leq 0.$$

Thus V is a subsolution, smaller than U. Therefore there exists a solution u_b such that $V \leq u \leq U$. This implies that $tr_{\mathbb{R}^N}(u_b) = \nu_\infty$. If $b' > b$ we construct $u_{b'}$ with $tr_{\mathbb{R}^N}(u_{b'}) = \nu_\infty$ and $\lim_{t\to\infty}(u_{b'}(0,t) - u_b(0,t)) > 0$. □

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Trace au bord de solutions d'équations de Hamilton-Jacobi elliptiques et trace initiale de solutions d'équations de la chaleur avec absorption sur-linéaire

Résumé. Cette thèse est constituée de trois parties. Dans la première partie, on s'intéresse au problème de *trace au bord* d'une solution positive de l'équation (E1) $-\Delta u + g(|\nabla u|) = 0$ dans un domaine borné Ω. Si $g(r) \geq r^q$ avec $q > 1$, on prouve que toute solution positive de (E1) admet une trace au bord considérée comme une mesure de Borel régulière. Si $g(r) = r^q$ avec $1 < q < q_c = \frac{N+1}{N}$, on montre l'existence d'une solution positive dont la trace au bord est une mesure de Borel régulière. Si $g(r) = r^q$ avec $q_c \leq q < 2$, on établit une condition nécessaire de résolution en terme de capacité de Bessel $C_{\frac{2-q}{q},q'}$. On étudie aussi des ensembles éliminables au bord pour des solutions modérées et sigma-modérées. La deuxième partie est consacrée à étudier la limite, lorsque $k \to \infty$, de solutions d'équation $\partial_t u - \Delta u + f(u) = 0$ dans $\mathbb{R}^N \times (0,\infty)$ avec donnée initiale $k\delta_0$. On prouve qu'il existe essentiellement trois types de comportement possible et démontre un résultat général d'existence de *trace initiale* et quelques résultats d'unicité et de non-unicité de solutions dont la donnée initiale n'est pas bornée. Dans la troisième partie, on considère l'équation $\partial_t u - \Delta_p u + f(u) = 0$ dans $\mathbb{R}^N \times (0,\infty)$ où $p > 1$. Si $p > \frac{2N}{N+1}$, on fournit une condition suffisante portant sur f pour l'existence et l'unicité des solutions fondamentales et on étudie la limite lorsque $k \to \infty$. On donne aussi de nouveaux résultats de non-unicité de solutions avec donnée initiale non bornée. Si $p \geq 2$, on prouve que toute solution positive admet une trace initiale dans la classe des mesures de Borel régulières positives. Finalement on applique les résultats ci-dessus au cas $f(u) = u^\alpha \ln^\beta(u+1)$ avec α, $\beta > 0$.

Mots clés. équations elliptiques quasilinéaires, singularités isolées, mesures de Radon, mesures de Borel, capacités de Bessel, trace au bord, singularités éliminables, absorption faiblement sur-linéaire, trace initiale, condition de Keller-Osserman, équations de la chaleur dégénérées.

Abstract. This thesis is divided into three parts. In the first part, we study the *boundary trace* of positive solutions of the equation (E1) $-\Delta u + g(|\nabla u|) = 0$ in a bounded domain Ω. When $g(r) \geq r^q$ with $q > 1$, we prove that any positive function of (E1) admits a boundary trace which is an outer regular Borel measure. When $g(r) = r^q$ with $1 < q < q_c = \frac{N+1}{N}$, we prove the existence of a positive solution with a general outer regular Borel measure as boundary trace. When $g(r) = r^q$ with $q_c \leq q < 2$, we establish a necessary condition for solvability in term of the Bessel capacity $C_{\frac{2-q}{q},q'}$. We also study boundary removable sets for moderate and sigma-moderate solutions. The second part is devoted to investigate the limit, when $k \to \infty$, of the solutions of $\partial_t u - \Delta u + f(u) = 0$ in $\mathbb{R}^N \times (0,\infty)$ with initial data $k\delta_0$. We prove that there exist essentially three types of possible behaviour and provide a new and more general construction of the *initial trace* and some uniqueness and non-uniqueness results for solutions with unbounded initial data. In the third part, we consider the equation $\partial_t u - \Delta_p u + f(u) = 0$ in $\mathbb{R}^N \times (0,\infty)$ where $p > 1$. If $p > \frac{2N}{N+1}$, we provide a sufficient condition on f for existence and uniqueness of the fundamental solutions and we study their limit when $k \to \infty$. We also give new results dealing with non uniqueness for the initial value problem with unbounded initial data. If $p \geq 2$, we prove that any positive solution admits an initial trace in the class of positive Borel measures. Finally we apply the above results to the case $f(u) = u^\alpha \ln^\beta(u+1)$ with α, $\beta > 0$.

Key words. quasilinear elliptic equations, isolated singularities, Radon measures, Borel measures, Bessel capacities, boundary trace, removable singularities, weakly superlinear absorption, initial trace, Keller-Osserman condition, degenerate heat equations.