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SURFACES MINIMALES DANS DES VARIÉTÉS HOMOGÈNES

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Résumé

Le cadre de cette thèse est la théorie des surfaces minimales dans deux variétés homogènes, $\mathbb{R}^3$ et $\widetilde{PSL_2}(\mathbb{R})$. Dans $\mathbb{R}^3$, étant donné un pavage $\mathcal{T}$ du plan par des polygones, qui soit invariant par deux translations indépendantes, on construit une famille de surfaces minimales plongées et triplement périodiques qui désingularise $\mathcal{T} \times \mathbb{R}$. Dans cette perspective, et inspiré par le travail de Martin Traizet, nous ouvrons les *nodes* d'une surface de Riemann singulière dans le but de coller ensemble des *Karcher saddle towers*, chacune placée sur un sommet avec ses bouts au long des arrêtes qui se terminent sur ce sommet même. Dans une seconde partie, nous étudions les graphes minimaux dans $\widetilde{PSL_2}(\mathbb{R})$ et nous fournissons des exemples de surfaces invariantes. Nous obtenons des estimées du gradient pour les solutions de l'équation des surfaces minimales dans l'espace en considération et on étudie le comportement des suites monotones de solutions. Nous concluons par prolonger à $\widetilde{PSL_2}(\mathbb{R})$ un théorème de Jenkins et Serrin, qui donnent une condition nécessaire et suffisante pour la solvabilité du problème du Dirichlet de l'équation des surfaces minimales dans $\mathbb{R}^3$, avec des données infinies sur le bord d'un domaine convexe et borné.

Mots clés : Variétés Homogènes simplement connexes de dimensions trois, Fibrations Riemannienne, Sections minimales, Surfaces minimales invariantes dans $\widetilde{PSL_2}(\mathbb{R})$, Théorème de type Jenkins-Serrin, Surfaces minimales triplement périodiques, Surfaces de Riemann singuliere, différentielles regulière, Karcher Saddle towers, Pavage rigide du plan.

Abstract

This doctoral thesis deals with minimal surface theory in two homogeneous manifolds, namely, $\mathbb{R}^3$ and $\widetilde{PSL_2}(\mathbb{R})$. In $\mathbb{R}^3$, given a tiling $\mathcal{T}$ of the plane by straight edge polygons, which is invariant by two independent translations, we construct a family of embedded triply periodic minimal surfaces which desingularizes $\mathcal{T} \times \mathbb{R}$. For this purpose, inspired by the work of Martin Traizet, we open the nodes of singular Riemann surfaces to glue together simply periodic Karcher saddle towers, each placed at a vertex of the tiling in such a way that its wings go along the corresponding edges of the tiling ending at that vertex. On the other hand, in $\widetilde{PSL_2}(\mathbb{R})$ we study minimal graphs and we furnish many invariant examples. We derive gradient estimates for solutions of the minimal surface equation in the underlying space and we study convergence of monotone sequences of solutions. Finally, we extend to $\widetilde{PSL_2}(\mathbb{R})$ a result of Jenkins and Serrin who provide a necessary and sufficient condition for the solvability of the Dirichlet problem of the minimal surface equation in $\mathbb{R}^3$, with infinite data over boundary arcs of a convex bounded region.

Keywords : Homogeneous simply connected 3-manifolds, Riemannian fibrations, Minimal sections, Invariant minimal surfaces in $\widetilde{PSL_2}(\mathbb{R})$, Jenkins-Serrin type theorem, Triply periodic minimal surfaces, Riemann surfaces with nodes, Regular differentials, Karcher saddle towers, Rigid planar tilings.

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Introduction

L'étude des surfaces minimales date du 18ième siècle et trouve ses racines dans les travaux de Lagrange, qui cherchait des graphes qui admettent comme bord une courbe fixe, et qui minimisent l'aire. Les difficultés rencontrées en essayant de comprendre ces surfaces, faisant appel à des branches différentes des mathématiques, ont nécessité des efforts de la part de plus grands mathématiciens du 19ième et du 20ième siècle. Une surface minimale est une surface dont la courbure moyenne est nulle en chacun de ses points. Egalement, et plus intuitivement, une surface minimale est telle que chacun de ses points admet un voisinage qui minimise l'aire par rapport à son bord.

Ce mémoire de doctorat est consacré à l'étude de la théorie des surfaces minimales dans deux variétés homogènes et simplement connexes de dimension trois, l'espace euclidien $\mathbb{R}^3$ et $\widetilde{PSL_2}(\mathbb{R})$, le revêtement universel du group linéaire spécial projectif d'ordre deux. Nous avons deux objectifs, d'un côté nous initions l'étude des surfaces minimales dans $\widetilde{PSL_2}(\mathbb{R})$, nous fournissons des exemples de surfaces invariantes par des groups à un paramètre d'isométries de $\widetilde{PSL_2}(\mathbb{R})$, et nous développons une machinerie pour montrer un théorème de type Jenkins-Serrin pour les graphes minimaux dans $\widetilde{PSL_2}(\mathbb{R})$. De l'autre côté, nous construisons une pléthore d'exemples de surfaces minimales triplement périodiques en collant ensembles des surfaces minimales simplement périodiques.

La variété $\widetilde{PSL_2}(\mathbb{R})$ porte d'une manière naturelle une des huit géométries maximales possibles sur une variété simplement connexe de dimension trois, comme décrites par Thurston. En fait, il y a un intérêt croissant dans l'étude de la théorie des surface minimales dans des espaces comme $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ et le group d'Heisenberg $Nil(3)$, chacun portant une des géométries dans le sens de Thurston, et donc il est naturel d'initier la théorie dans $\widetilde{PSL_2}(\mathbb{R})$. Il se trouve que $\widetilde{PSL_2}(\mathbb{R})$ est une fibration Riemannienne sur le plan hyperbolique $\mathbb{H}^2$, ce qui permet de considérer les graphes comme des sections de la projection du fibré. Le groupe d'isométries de $\widetilde{PSL_2}(\mathbb{R})$ est de dimension quatre et est engendré par les rélèvements des isométries de $\mathbb{H}^2$ et les translations le long des fibres verticales de $\widetilde{PSL_2}(\mathbb{R})$. Nous commençons notre investigation par le calcul de l'équation des surfaces minimales dans $\widetilde{PSL_2}(\mathbb{R})$ et nous l'utilisons pour trouver des exemples de surfaces minimales invariantes. Suivant les pas de Jenkins et Serrin dans $\mathbb{R}^3$ et ceux de Rosenberg et Nelli dans $\mathbb{H}^2 \times \mathbb{R}$, nous développons la machinerie nécessaire pour montrer un théorème de type Jenkins-Serrin dans $\widetilde{PSL_2}(\mathbb{R})$.

Notre théorème de type Jenkins-Serrin donne une condition nécessaire et suffisante pour la solvabilité du problème du Dirichlet de l'équation des surface minimales dans $\widetilde{PSL_2}(\mathbb{R})$, avec des données infinies prescrites sur des arcs du bord d'un domaine convexe et borné de $\mathbb{H}^2$, et avec une donnée continue sur le reste du bord. Cependant, un arc du bord d'un domaine de $\mathbb{H}^2$, où une solution de l'équation des surfaces minimales dans $\widetilde{PSL_2}(\mathbb{R})$ admet une valeur infinie, doit être géodésique.

Plus précisément, soit Ω un domaine convexe et borné dans $\mathbb{H}^2$ dont le bord consiste en des arcs géodésiques (ouverts) $A_1, .., A_n, B_1, ..., B_m$ et leurs extrémités, et des arcs ouverts convexes $C_1, C_2, ..., C_\ell$. On suppose que les géodésiques A_i (resp. B_i) n'admettent pas d'extrémité commune. Nous fournissons une condition nécessaire et suffisante pour l'existence d'une section minimale $s : \Omega \to \widetilde{PSL_2}(\mathbb{R})$ de la submersion Riemannienne, qui admet la valeur $+\infty$ sur les arcs $A_1, ...A_n$, $-\infty$ sur les arcs $B_1, ..., B_m$, et une donnée continue arbitrairement prescrite sur les arcs C_i. Pour un polygone géodésique simple $\mathcal{P}$, dont les sommets sont choisis parmi les extrémités des A_i et B_j, nous désignons par α et β les $\mathbb{H}^2$-longueurs totales des segments A_i et les segments B_j respectivement qui font partie de $\mathcal{P}$. Nous remarquons que dans le cas $\{C_s\} = \emptyset$, $\mathcal{P}$ peut être le bord de Ω tout entier. Nous énonçons notre resultat principal dans $\widetilde{PSL_2}(\mathbb{R})$

Théorème *Si la famille $\{C_s\} \neq \emptyset$, il existe alors une section unique du fibré $\pi : \widetilde{PSL_2}(\mathbb{R}) \to \mathbb{H}^2$ définie dans Ω qui admet la valeur $+\infty$ sur les arcs géodésiques A_i, la valeur $-\infty$ sur les géodésiques B_j et une donnée continue arbitraire f_s sur C_s si et seulement si*

$$2\alpha < \gamma \textit{ and } 2\beta < \gamma$$

pour chaque polygone $\mathcal{P}$ choisi comme ci-dessus.
Si la famille $\{C_s\} = \emptyset$, la condition sur les polygones $\mathcal{P}$ est la même sauf dans le cas ou $\mathcal{P}$ est le bord de Ω oú la condition devient $\alpha = \beta$. De plus l'unicité est à une constante près.

Un résultat important pour notre travail est l'existence d'une solution du problème de Dirichlet de l'équation des surfaces minimales dans $\widetilde{PSL_2}(\mathbb{R})$, dans un domaine convexe et borné de $\mathbb{H}^2$ avec une donnée continue par morceaux. Important aussi est le comportement des suites monotone de solutions dans ce genre de domaine. Dans leur travail, Jenkins et Serrin emploient des estimées a priori pour les solutions de l'équation des surfaces minimales dans $\mathbb{R}^3$, et ils utilisent la surface de Scherk comme barrière, ce qui est fondamental pour la plupart des résultats. Les techniques de Jenkins-Serrin ont été adaptées dans $\mathbb{H}^2 \times \mathbb{R}$ par Rosenberg et Nelli, qui construisent une surface de type Scherk dans cet espace. Cependant, ces techniques ne se prolongent pas d'une manière evidente dans $\widetilde{PSL_2(\mathbb{R})}$, qui n'est pas un éspace produit ! Nous adaptons des résultats de Spruck pour obtenir une estimée de gradient pour les solutions dans l'espace en considération, ce qui implique un principe de compacité. Ensuite nous étudions le comportement des suites monotones de solutions et nous suivons les constructions de Jenkins et Serrin pour établir notre théorème.

Nous poursuivrons avec une description de notre construction de surfaces minimales plongées triplement périodiques. Au cours du 19ième siècle, H.A. Schwarz explorait les surfaces minimales périodiques et il a pu construire cinq surfaces triplement périodiques. Une surface minimale dans l'espace euclidien est dite triplement périodique, si elle est invariante par trois translations indépendantes. Sa méthode consistait à refléter une surface minimale de type disque, bordée par une ligne polygonale non plane, par rapport aux segments du bord. Dans les années 1970, le physicien et cristallographe Alan Schoen, a découvert plusieurs exemples de surfaces minimales triplement périodiques et il en a construit des modèles. Quelques années plus tard, Herman Karcher a pu établir rigoureusement l'existence des surfaces de Schoen, et il a pu construire de nouvelles familles de surfaces minimales triplement périodiques en appliquant sa méthode "Conjugate plateau constructions".

Nous construisons des familles de surfaces minimales triplement périodiques en collant ensemble des surfaces minimales simplement périodiques avec des bouts de type Scherk. Un bout de type Scherk est asymptotique a un demi-plan. Heuristiquement, l'idée derrière le processus de construction est la suivante : de loin, une *Karcher saddle tower* se voit comme une collection de demi-plans verticaux ayant une ligne en commun. Nous supposons que tout les saddle towers admettent la même période verticale $(0, 0, T)$, qu'on normalise à $(0, 0, 2\pi)$. Nous rapetissons les saddle towers d'un facteur ε^2, de façon que la période verticale soit $P_\varepsilon = (0, 0, \varepsilon^2 T)$. Alors étant donné un pavage de $\mathbb{R}^2$, invariant par deux translation indépendantes, nous plaçons une *Karcher saddle tower* rapetissée sur chaque sommet de $\mathcal{T}$ de telle manière que le nombre de bouts de la *saddle tower* soit égal au nombre des arêtes qui se terminent au sommet, et chaque bout va le long d'une arête. Pour chaque arête nous collons les bouts des *saddle towers* placées sur ses extrémités, ce qui va donner une surface minimale triplement périodique d'une période horizontale celle du pavage et d'une période verticale P_ε. Il est naturel qu'on suppose les surfaces à construire symétriques par rapport au plan horizontal car les *Karcher saddle towers* le sont. Bien sûr, nous n'attendons pas que la construction marche pour des pavages périodiques arbitraires. Par exemple, il sera nécessaire d'avoir des pavages qui soient équilibrés (au sens *balanced* de notre article) pour qu'on puisse placer les *Karcher saddle towers* comme nous l'avons expliqué. Nous montrerons qu'il suffit d'avoir des pavages orientables, équilibrés et rigide pour que tout marche bien. Notre résultat se résume dans le théorème suivant

Théorème Soit $\mathcal{T}$ un pavage équilibré du plan qui est invariant par deux translations indépendantes T_1 et T_2. Soit Γ le groupe engendré par T_1 et T_2, et $\mathcal{T}$ le pavage correspondant dans $\mathbb{R}^2/\Gamma$.

Si dans le quotient $\mathcal{T}$ est orientable et rigide alors pour chaque $\varepsilon \neq 0$ suffisamment petit, il existe une surface minimale triplement périodique M_ε d'une période horizontale Γ et d'une periode verticale $(0, 0, 2\pi\varepsilon^2)$ telle que :

1. M_ε est symétrique par rapport au plan horizontal et dépend de ε d'une manière continue.
2. Quand $\varepsilon \to 0$, M_ε converge, sur les compacts de $\mathbb{R}^3$ et pour la métrique ambiante, vers l'ensemble $\mathcal{T} \times \mathbb{R}$.
3. Autour de chaque sommet v de $\mathcal{T}$ dans le plan, M_ε agrandi d'un facteur ε^{-2}, ressemble à une *Karcher saddle tower* M_v, d'une période verticale $(0, 0, 2\pi)$, dont les bouts sont en bijection avec les arêtes qui se terminent en v d'une manière que le demi-plan vertical asymptotique à un bout de M_v soit parallèle à son arête correspondante et pointe dans la même direction.

   Plus précisément, il existe un vecteur horizontal ν_ε tel que $\varepsilon^{-2}(M_\varepsilon - \nu_\varepsilon)$ converge vers $M_v$sur les compacts de $\mathbb{R}^3$.

La construction sera menée en fournissant les données de Weierstrass sur une surface de Riemann bien adaptée. Inspiré par le travail de Martin Traizet, nous ouvrons les *nodes* d'une surface de Riemann singulière oú nous exploitons la régénération des formes régulières qu'elles portent vers des formes holomorphes. Nous ajusterons les paramètres qui sous-tendent la construction en appliquant le théorème des fonctions implicites.

Introduction

The study of minimal surfaces dates back to the late 18-th century, and is rooted in the work of Lagrange who sought graphs with least area, spanning particular space curves. The difficulty encountered in understanding these surfaces and the interplay between different branches of mathematics employed through their study, engaged the efforts of many of the greatest mathematicians of the 19-th and the 20-th century. A minimal surface is one whose mean curvature is identically equal to zero. Equally and more appealing to intuition, a minimal surfaces is one which each of its points admits a neighborhood of least area with respect to its boundary.

The doctoral thesis at hand is devoted to the study of minimal surface theory in two homogeneous and simply connected Riemannian 3-manifolds, namely, the euclidean three space $\mathbb{R}^3$ and $\widetilde{PSL_2}(\mathbb{R})$, the universal covering of the projective real special linear group of order two. Our objective is two-fold, on the one hand we initiate the study of minimal surfaces in $\widetilde{PSL_2}(\mathbb{R})$, we furnish examples of minimal surfaces invariant under one parameter groups of isometries of $\widetilde{PSL_2}(\mathbb{R})$, and we develop the machinery culminating to a proof of a Jenkins-Serrin type theorem for minimal graphs in $\widetilde{PSL_2}(\mathbb{R})$. On the other hand, we furnish a plethora of examples of triply periodic minimal surfaces embedded in the euclidean space by gluing together simply periodic minimal surfaces.

The manifold $\widetilde{PSL_2}(\mathbb{R})$ carries naturally one of the eight maximal geometries possible on simply connected three manifolds, as described by Thurston. In fact, in recent years there has been an increasing interest in the study of minimal surface theory in spaces like $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and the Heisenberg group $Nil(3)$, each carrying a different maximal geometry in the sense of Thurston, and it was only natural to initiate the theory in $\widetilde{PSL_2}(\mathbb{R})$. It turns out that $\widetilde{PSL_2}(\mathbb{R})$ is a Riemannian fibration over the hyperbolic plane $\mathbb{H}^2$, which permits us to deal with graphs in this space as sections of the fiber bundle projection. The isometry group of $\widetilde{PSL_2}(\mathbb{R})$ is four dimensional and is generated by lifts of the isometries on $\mathbb{H}^2$ and translations along the vertical fibers of $\widetilde{PSL_2}(\mathbb{R})$. We start our investigation by deriving the minimal surface equation in $\widetilde{PSL_2}(\mathbb{R})$ and we furnish examples of invariant minimal surfaces. Following the lines of work of Jenkins and Serrin in $\mathbb{R}^3$ and those of Rosenberg and Nelli in $\mathbb{H}^2 \times \mathbb{R}$, we then develop the machinery necessary to prove a Jenkins-Serrin type theorem in $\widetilde{PSL_2}(\mathbb{R})$.

Our Jenkins-Serrin type theorem gives necessary and sufficient conditions for the solvability of the Dirichlet problem for the minimal surface equation in $\widetilde{PSL_2}(\mathbb{R})$, allowing infinite boundary values prescribed on arcs of the boundary of a convex bounded domain in $\mathbb{H}^2$, and continuous data on the rest of the boundary. However, boundary arcs of a bounded domain in $\mathbb{H}^2$, where a solution of the minimal surface equation in $\widetilde{PSL_2}(\mathbb{R})$ admits infinite values, have to be geodesics.

Then more precisely, let Ω be a convex bounded domain in $\mathbb{H}^2$ whose boundary consists of (open) geodesic arcs $A_1, .., A_n, B_1, ..., B_m$, together with their end points and convex open arcs $C_1, C_2, ..., C_s$. We suppose that no two geodesics A_i and no two geodesics B_i have a common end point. We give necessary and sufficient conditions for the existence of a minimal section $s : \Omega \to \widetilde{PSL_2}(\mathbb{R})$ of the Riemannian submersion, taking values $+\infty$ on the arcs $A_1, ...A_n$, $-\infty$ on the arcs $B_1, ..., B_m$ and arbitrary prescribed continuous data on the arcs $C_1, ..., C_s$. For a simple closed geodesic polygon $\mathcal{P}$, whose vertices are chosen from among the endpoints of the segments A_i and the segments B_j, let α and β be, respectively, the total $\mathbb{H}^2$-length of the geodesics A_i and the total $\mathbb{H}^2$-length of the geodesics B_i which are part of $\mathcal{P}$. Let γ be the perimeter of $\mathcal{P}$. Note that in the case $\{C_s\} = \emptyset$, $\mathcal{P}$ could be the whole boundary of Ω. We can now state our principal result in $\widetilde{PSL_2}(\mathbb{R})$,

Theorem. *If the family of arcs $\{C_s\}$ is non empty, then there exists a unique section of the bundle $\pi : \widetilde{PSL_2}(\mathbb{R}) \to \mathbb{H}^2$ defined in Ω and taking the boundary values $+\infty$ on the geodesics A_i, the value $-\infty$ on the geodesics B_i and arbitrary continuous data f_s on C_s if and only if*

$$2\alpha < \gamma \text{ and } 2\beta < \gamma$$

for each polygon $\mathcal{P}$ chosen as above.
If the family of arcs $\{C_s\}$ is empty, the condition on the polygons $\mathcal{P}$ is the same except that in the case when $\mathcal{P}$ is the entire boundary of Ω then the condition is $\alpha = \beta$. Moreover, uniqueness is up to additive constants.

Fundamental to our work in $\widetilde{PSL_2}(\mathbb{R})$ is the existence of a solution of the Dirichlet problem, associated to the minimal surface equation in $\widetilde{PSL_2}(\mathbb{R})$, in a convex bounded domain of $\mathbb{H}^2$ with piecewise continuous boundary data. Fundamental also is the behavior of monotone sequences of solutions in such domains. In their paper, Jenkins and Serrin make use of the a priori estimates for solutions of the minimal surface equation to obtain a compactness principle for sequences of solutions and to study limit behavior of monotone sequences of solutions. They also make use of the Scherk surface as a barrier, which is fundamental to most of the results. The techniques developed by Serrin were adapted by Rosenberg and Nelli to show a Jenkins-Serrin type theorem in $\mathbb{H}^2 \times \mathbb{R}$, where the authors construct a Scherk type surface. However, these techniques do not extend in an obvious way to $\widetilde{PSL_2}(\mathbb{R})$, which is not a product space. To prove our corresponding compactness principle and to develop necessary tools to study limit behavior of monotone sequences of solutions, we adapt results of Spruck and we construct barriers adequate to our space.

We next describe our construction of embedded triply periodic minimal surfaces. During the middle of the nineteenth century, H.A. Schwarz carried an intensive investigation of periodic minimal surfaces and was able to construct five triply periodic ones. A minimal surface in the euclidean space is said to be triply periodic if it is invariant under three independent translations. His method consisted of spanning a disc-type minimal surface into a non-planar polygonal boundary, and then reflecting this surface across its boundary lines.

In the 1970's, the physicist and crystallographer Alan Schoen, discovered many triply periodic minimal surfaces and constructed models of them. However, his study of these surfaces was a bit sketchy and thus, among mathematicians, there remained doubts whether all details could be filled in. It did not take long until Hermann Karcher established rigorously the existence of all of Schoen's surfaces, and constructed whole families of newly found triply periodic embedded minimal surfaces, by applying his so called "Conjugate Plateau Constructions" .

We will construct families of triply periodic embedded minimal surfaces by gluing simply periodic ones with Scherk type ends. A Scherk type end is one which is asymptotic to a vertical half-plane. Roughly speaking, the idea underlying the gluing process is the following : from a distance, a Karcher saddle tower is seen as a set of vertical half-planes intersecting at a common line. We assume that all of the saddle towers to be considered, admit the same vertical period $(0, 0, T)$, which without loss of generality we normalize to $(0, 0, 2\pi)$. We scale each of the saddle towers by a factor of ε^2 (!), so that the vertical period is $P_\varepsilon = (0, 0, \varepsilon^2 T)$. Then, given a tiling $\mathcal{T}$ of $\mathbb{R}^2$, invariant under two independent translations, we place a scaled Karcher saddle tower at each of its vertices in such a way that, the number of wings of the saddle tower is equal to the number of edges ending at the vertex where it is placed, and each wing goes along an edge. For each edge we glue the corresponding wings of the scaled saddle towers placed at its ends, resulting in a triply periodic surface whose horizontal period is that given by the tiling and a vertical period P_ε. It is natural that we require the surfaces to be symmetric with respect to the horizontal plane as it is the case for the saddle towers under consideration. Of course, we do not expect the construction to work for arbitrary periodic tilings. For example, it will be necessary to have a balanced tiling as to place the Karcher saddle towers at the different vertices as explained above. It turns out that, for our purposes, rigid and balanced tilings will do. In our work, we establish criteria to help decide the rigidity of a given tiling. We resume our construction as follows,

Theorem *Let $\mathcal{T}$ be a balanced tiling in the plane which is invariant by two independent translations T_1 and T_2. Let Γ denote the group generated by T_1 and T_2, and denote by $\mathcal{T}$ the corresponding quotient tiling in $\mathbb{R}^2/\Gamma$.*

If the quotient tiling $\mathcal{T}$ is orientable and rigid then for any $\varepsilon \neq 0$ sufficiently small, there exists an embedded triply periodic minimal surface M_ε with horizontal period Γ and a vertical period $(0, 0, 2\pi\varepsilon^2)$ such that :

1. *M_ε is symmetric with respect to the horizontal plane and depends continuously on ε.*
2. *When $\varepsilon \to 0$, M_ε converges, on compact subsets of $\mathbb{R}^3$ and for the ambient metric, to the set $\mathcal{T} \times \mathbb{R}$.*
3. *In a neighborhood of each vertex v of $\mathcal{T}$ in the plane, when scaled by ε^{-2}, M_ε looks like a Karcher saddle tower M_v whose period is equal to $(0, 0, 2\pi)$, and whose ends are in a one-to-one correspondence with the edges ending at v in such a way that, the asymptotic vertical half-plane to an end of M_v is parallel to its corresponding edge and points in the same direction.*

   More precisely, for each v there exists a horizontal vector ν_ε such that, $\varepsilon^{-2}(M_\varepsilon - \nu_\varepsilon)$ converges to M_v on compact subsets of $\mathbb{R}^3$.

The construction will be accomplished by furnishing Weierstrass data on appropriate Riemann surfaces, where we employ Weierstrass representation of a minimal surface in its simplest form. Inspired by the work of Martin Traizet, we perform the gluing by opening the nodes of singular Riemann surfaces with nodes and we invest the regeneration of the regular differential forms they carry into holomorphic forms. We adjust the parameters underlying the construction underlying the construction by applying the implicit function theorem.

Première partie

Théorie de Base

Chapitre 1

Minimal Surfaces

1.1 Notions on surfaces

For our purposes, a surface $\mathcal{S}$ in $\mathbb{R}^3$ is considered as the image of a 2-manifold Σ by an immersion $X : \Sigma \to \mathbb{R}^3$. X defines a local embedding on Σ and therefore when we treat local questions on $\mathcal{S}$, we identify Σ and $\mathcal{S}$ and confound them. Σ inherits a metric from $\mathbb{R}^3$ as follows : if U and V are tangent vectors to Σ

$$\langle U, V\rangle = \langle DX(U), DX(V)\rangle_{\mathbb{R}^3}.$$

The Levi-Civita ∇ connection corresponding to the metric induced on Σ is then defined by the following relation : if U and V are vector fields on Σ

$$DX(\nabla_U V) = [U(DX(V)]^T,$$

where $[.]^T$ denotes projection onto the plane $D_pX(T_p\Sigma)$ tangent to $\mathcal{S}$ at $X(p)$, $p \in \Sigma$. Note that $U(DX(V))$ is equal to the covariant derivative in $\mathbb{R}^3$ of $DX(V)$ by $DX(U)$. We restrict our attention to orientable surfaces and we assume that an orientation is fixed on $\mathcal{S}$. We define the Gauss map N of Σ, $N : \Sigma \to \mathbb{S}^2$, as follows : for $p \in \Sigma$ let $N(p)$ be the unit normal to $\mathcal{S}$ at $X(p)$. Since $N(p)$ is normal to both $T_{N(p)}\mathbb{S}^2$ and $T_p\mathcal{S}$, the two linear spaces may be identified and the differential $A_p = D_pN$ then defines a self-adjoint endomorphism of $T_p\Sigma$.
A is called the shape operator of Σ and it verifies

$$-\langle A(U), V\rangle = U(DX(V)).N \ ,$$

an equation which can be simply obtained from $DX(V).N = 0$ by taking the $\mathbb{R}^3$ covariant derivative by U. The left side term of this equation is the second fundamental form of Σ.
The eigenvalues of A_p, $p \in \Sigma$, and their corresponding eigenvectors are called respectively the principal curvatures and the principal directions of Σ at p. The average of the principal curvatures is the mean curvature of $\mathcal{S}$ (or X, or Σ!) which we denote by H.

Definition 1 *A minimal surface in* $\mathbb{R}^3$ *is one whose mean curvature vanishes at each of its points.*

Remark 1 *The Gauss map and the mean curvature of* Σ *are defined via the immersion* X *into* $\mathbb{R}^3$*. These quantities may be defined in a similar manner for oriented hypersurfaces immersed in an oriented Riemannian manifold, and one may therefore speak of a minimal surface in oriented Riemannian 3-manifolds.*

1.2 Minimal Graphs

It is generally admitted that the study of minimal surfaces started with Lagrange in the 1760's, when he addressed the following minimization problem : Given a bounded domain Ω in the plane and a continuous function g defined on its boundary $\partial\Omega$, what smooth functions u defined in Ω and admitting the values g on $\partial\Omega$ have their graphs with the least area. One has the following standard formula (see [19]) :

Proposition 1 *(First Variation Formula). Let* $f : M \to \mathbb{R}^3$ *be a compact surface with boundary. Let* $f_t : M \to \mathbb{R}^3$ *be a smooth variation of* f *for* $t \in (-1, 1)$ *such that* $f_0 = f$*. Let* $E = f_*(\frac{d}{dt})$ *be the variational field restricted to* $f_0 = f$*. If* $\mathcal{A}(t)$ *is the area of* f_t *then*

$$\mathcal{A}'(0) = -2 \int_M \langle E, H.N \rangle \, d\mathcal{A} \tag{1.1}$$

where H *is the mean curvature of* M*,* N *its Gauss map and* dA *is the area form of the metric induced by* f*.*

Remark 2

(i) Proposition 1 characterizes minimal surfaces as being the critical point of the area functional.

(ii) (Second Variation Formula) If we consider the variations $f_t = f + t\phi N$*, where* ϕ *is a smooth function of compact support away from the boundary, we can prove that*

$$\mathcal{A}''(0) = -\int_M \phi(\Delta\phi - 2K\phi) d\mathcal{A}, \tag{1.2}$$

where K *is the Gaussian curvature of* M *and* Δ *its Laplacian, see [19].*

An immediate consequence of Proposition 1 is that u is a solution of the above minimization problem only if the graph of u has zero mean curvature at interior points of Ω, *i.e.* a piece of minimal surface. However, for a surface given by $F(x, y, z) = z - u(x, y) = 0$ the mean curvature with respect to the upward pointing normal can be simply computed to obtain

$$H = -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right).$$

Then $H \equiv 0$ could be written into the quasilinear, second order, elliptic partial differential equation

$$(1 + u_x^2)u_{yy} + 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0 \tag{1.3}$$

known as the minimal surface equation. Therefore, a solution of the minimization problem is necessarily the solution of a Dirichlet problem in Ω and when Ω is convex we have the following existence and uniqueness result which allows g to admit a finite number of discontinuities on $\partial\Omega$.

Theorem 1 *Let Ω be a bounded convex domain in the plane. Consider a finite set of points in $\partial\Omega$, and let C denote the remaining boundary of Ω (which consists of a finite number of open arcs). Then there exists a solution of the minimal surface equation in Ω, taking on preassigned bounded continuous data on the arcs C.*

It should be noted that the restriction to convex domains is necessary in order that solutions exist corresponding to arbitrarily given piecewise continuous boundary data. One asks whether a solution u of (1.3) solves the minimization problem. The answer is positive if $\mathcal{A}''(0) > 0$ and elliptic differential equations theory says its always the case for minimal graphs spanning compact curves in $\mathbb{R}^3$. We say that minimal graphs are stable. More generally, a minimal surface M is said to be stable if for any of its relatively compact domains $\mathcal{O}$, the Dirichlet problem associated to the Jacobi operator $L = \Delta - 2K = \Delta + |A|^2$ in $\mathcal{O}$ has no negative eigenvalues. For orientable minimal surfaces, stability is equivalent to the existence of a positive Jacobi function on M (see [13]). For minimal graphs over planar domains, the Gauss map N has its image set contained in an open half-sphere, and the inner product of N with the unit normal to the plane of the domain provides a positive Jacobi function, from which we conclude that any minimal graph is stable. In fact, any surface can be expressed as a graph over a domain of the tangent plane around each of its points. After a rotation, a minimal surface can be locally expressed as a function $u = u(x, y)$ which verifies the minimal equation (1.3) and hence locally minimizes the area and it is where minimal surfaces derive their name.

Next we present a result which can be seen as an extension of Theorem 1. Let D denote the unit disc in the plane.

Theorem 2 *(Douglas, Rado)Let Γ be a rectifiable Jordan curve in $\mathbb{R}^3$, then there exists a unique minimal disc, $X : D \to \mathbb{R}^3$, of least area with $X(\partial D) = \Gamma$. If Γ admits a one-to-one orthogonal projection onto a convex curve γ in the plane, then the minimal disc is in fact a graph over the bounded region enclosed by γ.*

In fact the above theorem is a particular case of the well known Plateau problem, see [19] for example. As far as Theorem 1 is concerned, one may ask whether there exist with infinite boundary data admitted on arcs of the boundary on Ω as illustrated by the following example.

The Scherk Surface Let $\Omega = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and we consider the following solution of the minimal surface equation (1.3)

$$u(x, y) = -\ln(\cos x) + \ln(\cos y),\ (x, y) \in \Omega.$$

u takes the value $-\infty$ on the vertical boundary segments of Ω and the value $+\infty$ on the horizontal ones. In their paper [16], Jenkins and Serrin give a necessary and sufficient condition for the existence of a solution of the minimal surface equation (1.3) with infinite boundary data assigned on boundary arcs of a convex bounded domain, together with continuous data on the remaining part of the boundary. It turns out that the boundary arcs where the data is infinite must be straight line segments. We resume their result as follows.

Let Ω be a convex bounded domain of $\mathbb{R}^2$ whose boundary consists of open straight line segments A_1, A_2,...,A_k, B_1,B_2,...,B_l, open convex arcs C_1,C_2,...,C_s and their end points. We assume that no two segments A_i or B_j have a common end point. For a polygon P in Ω whose vertices are chosen from among those of the segments A_i and B_j, let α and β denote the total lengths of the segments A_i in P and the segments B_j in P respectively. We let γ denote the perimeter of P.

Theorem 3 *If the family of arcs $\{C_s\}$ is non empty, then there exists a unique solution of the minimal equation* (1.3) *defined in Ω and taking the boundary values $+\infty$ on the geodesics A_i, the value $-\infty$ on the geodesics B_i and arbitrary continuous data f_s on C_s if and only if*

$$2\alpha < \gamma \text{ and } 2\beta < \gamma$$

for each polygon $\mathcal{P}$ whose vertices are chosen from among those of the segments A_i and B_j.

If the family of arcs $\{C_s\}$ is empty, the condition on the polygons $\mathcal{P}$ is the same except that in the case when $\mathcal{P}$ is the entire boundary of Ω then the condition is $\alpha = \beta$. Moreover, uniqueness is up to additive constants.

For example, when Ω is the bounded region enclosed by a convex quadrilateral whose edges are A_1, C_1, A_2, C_2, in that order, then the condition in the above theorem reduces to $|A_1| + |A_2| < |C_1| + |C_2|$. If the edges of Ω were A_1, B_1, A_2, B_2 in that order, then the condition reduces to $|A_1| + |A_2| = |B_1| + |B_2|$. For domains Ω having at most one segment A_i and at most one segment B_j the condition of the theorem is trivially verified and the corresponding Dirichlet problem is always solvable. If Ω is bounded by a regular $2n$-gon, whose edges are alternately A_i and B_j, then the condition of the theorem holds for all the polygons P whose vertices are chosen from among those of the segments A_i and B_j, and the Dirichlet problem for these domains is always solvable.

Theorem 3 has been extended to (geodesically) convex and bounded domains Ω in the hyperbolic plane $\mathbb{H}^2$ bounded by open geodesic arcs A_i, open geodesic arcs B_j, convex arcs C_s and their endpoints. Rosenberg and Nelli [25] have given a necessary and sufficient condition, similar to that of Theorem 3, for the Dirichlet problem of the minimal surface equation in $\mathbb{H}^2 \times \mathbb{R}$ with infinite data $+\infty$ on the geodesic arcs A_i, $-\infty$ on the geodesic arcs B_j and continuous data on the convex ars C_s.

1.3 The Weierstrass representation

We have in $\mathbb{R}^3$ a powerful tool to construct and analyze examples of minimal surfaces. It turns out that each of these surfaces can be realized as the real part of a holomorphic curve, which allows complex analysis to come into play and the beautiful machinery of Riemann surfaces theory will be put into action. We show how to represent a minimal surface in $\mathbb{R}^3$ as such and we furnish some examples.

Let $\mathcal{S}$ be a minimal surface, whose position vector is $X : \Sigma \to \mathbb{R}^3$. We write $X = (x_1, x_2, x_3)$ and by ΔX we mean $(\Delta x_1, \Delta x_2, \Delta x_3)$, where Δ is the Laplacian with respect to metric X induces on Σ. If $(e_i)_{1\leq i\leq 2}$ is an orthonormal basis for $T_p\Sigma$, $p \in \Sigma$, we have the following

$$\Delta X = \sum (\nabla_{e_i} DX)(e_i),$$

with $DX = (dx_1, dx_2, dx_3)$. However for U and V tangent to Σ we have

$$\begin{aligned}
(\nabla_U DX)(V) &= U(DX(V)) - DX(\nabla_U V) \\
&= U(DX(V)) - [U(DX(V))]^T \\
&= (U(DX(V)).N)N \\
&= - \langle A(U), V \rangle N.
\end{aligned}$$

This shows that the $\Delta X = -tr(A)N$ and as the trace of an endomorphism is the sum of its eigenvalues we obtain the following formula $\Delta X = -2HN$. This preceding formula leads to drastic consequences on minimal surfaces as it shows that X, and thus each of the coordinate functions x, is in fact harmonic for the metric Σ inherits from $\mathbb{R}^3$.
Therefore each of the coordinate functions x gives rise, in a neighborhood of each $p \in \Sigma$ where $dx \neq 0$, to a conjugate harmonic function x^* so that $x + ix^*$ defines a conformal coordinate chart around p (knowing that isothermal parameters exist around each of the points of Σ). Since X is an immersion, *i.e.* $DX \neq 0$, we conclude that at each $p \in \Sigma$ at least one coordinate function satisfies $d_p x_i \neq 0$. Therefore, minimal surfaces in $\mathbb{R}^3$ inherit naturally, from their euclidean coordinates, an atlas of holomorphic functions. We think of Σ as a Riemann surface and we note that X maps Σ into $\mathbb{R}^3$ conformally. For each $p \in \Sigma$, the endomorphism A_p annihilates its characteristic polynomial giving that $A_p^2 - tr(A_p)A_p + det(A_p)Id = 0$, which shows that the Gauss map of Σ verifies

$$\langle DN(U), DN(V) \rangle = \kappa^2 \langle U, V \rangle$$

where $\pm\kappa$ are the principal curvatures. Therefore, N is anti-conformal. We orient the unit sphere by its outward pointing normal and we let σ denote the stereographic projection from the north pole to the complex plan identified with the (x_1, x_2)-plane. Then σ is orientation reversing and $g = \sigma \circ N : \Sigma \to \mathbb{C} \cup \{\infty\}$ is orientation-preserving and conformal whenever $DN \neq 0$. Thus g defines a meromorphic function on Σ. We have shown that a minimal surface carries a natural Riemann surface structure, with respect to which the Gauss map is meromorphic.
We set $X^* = (x_1^*, x_2^*, x_3^*)$, where x_j^* is the harmonic conjugate of x_j, and note that X^* is well defined locally. X^* is called the conjugate minimal immersion of X and it is well defined on some covering of Σ. The relation $dX^* = -dX \circ Rot_{\frac{\pi}{2}}$, which follows easily once we realize that $grad(x_j) = Rot_{\frac{\pi}{2}}(grad(x_j^*))$, shows that we have a globally defined holomorphic form $\Phi = dX + idX^*$ on Σ. Let $\Phi = (\phi_1, \phi_2, \phi_3)$, where $\phi_j = dx_j + idx_j^*$. We note that we may recover X as follows

$$X = Re \int \Phi = Re \int (\phi_1, \phi_2, \phi_3).$$

We next encode the properties of X into conditions on the holomorphic differentials ϕ_i. With respect to any local parameter $z = u_1 + iu_2$ on Σ,

$$\Phi = \Big(\frac{\partial X}{\partial u_1} + i\frac{\partial X^*}{\partial u_1}\Big)dz = \Big(\frac{\partial X}{\partial u_1} - i\frac{\partial X}{\partial u_2}\Big)dz.$$

Then

$$\begin{aligned}\Phi^2 &= \Big(\frac{\partial X}{\partial u_1} - i\frac{\partial X}{\partial u_2}\Big).\Big(\frac{\partial X}{\partial u_1} - i\frac{\partial X}{\partial u_2}\Big)dz^2\\ &= \Big(\left|\frac{\partial X}{\partial u_1}\right|^2 - \left|\frac{\partial X}{\partial u_2}\right|^2 - 2i\frac{\partial X}{\partial u_1}.\frac{\partial X}{\partial u_2}\Big)dz^2.\end{aligned}$$

The preceding formula then shows that X is conformal if and only if $\Phi^2 = 0$ which we write as

$$X \text{ is conformal if and only if } \phi_1^2 + \phi_2^2 + \phi_3^2 = 0 \text{ on } \Sigma.$$

A similar computation shows that since X is conformal

$$|\Phi|^2 = 2\left|\frac{\partial X}{\partial u_1}\right|^2 |dz|^2.$$

The metric, say ds^2, X induces on Σ can then be written as $ds^2 = \frac{1}{2}|\Phi|^2$, and the fact that X is a regular immersion encodes to $|\Phi| > 0$. We write this as follows

$$X \text{ is a regular immersion if and only if } |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0 \text{ on } \Sigma.$$

This representation of a minimal immersion as the integral of a holomorphic form verifying the above conditions is known as the Weierstrass representation of a minimal surface.

Remark 3 *Let* $dh = \phi_3 = dx_3 - idx_3 \circ Rot_{\frac{\pi}{2}}$. *Then it is not difficult to show that* $\phi_1 = (g^{-1} - g)\dfrac{dh}{2}$ *and* $\phi_2 = i(g^{-1} + g)\dfrac{dh}{2}$, *where* g *is the stereographic projection of the Gauss map as explained above (see [14]). The Weierstrass representation of a minimal surface becomes*

$$X = Re \int \big((g^{-1} - g)\frac{dh}{2}, i(g^{-1} + g)\frac{dh}{2}, dh\big). \tag{1.4}$$

The poles and zeros of g *coincide with the zeros of* dh *since the forms* ϕ *are holomorphic.*

What we have developed above suggests a recipe to construct minimal surfaces. We start with a given Riemann surface, say Σ, and we furnish three holomorphic differentials, say ϕ_1, ϕ_2 andϕ_3, verifying the conditions

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0 \text{ on } \Sigma \tag{1.5}$$

and

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0 \text{ on } \Sigma. \tag{1.6}$$

We set $X(p) = Re \displaystyle\int^p (\phi_1, \phi_2, \phi_3)$ and $\Phi = (\phi_1, \phi_2, \phi_3)$. Then X defines, up to a translation, a conformal harmonic regular immersion on Σ provided that we have

$$X = Re \int_\alpha \Phi = 0 \tag{1.7}$$

for all closed cycles α on Σ.
Note that since X is conformal, the conformal structure it induces on Σ (given by the metric inherited from $\mathbb{R}^3$) is compatible with that given on Σ in the first place. This implies that X is also harmonic for the metric it induces on Σ and hence defines a minimal immersion. The condition (1.7) is called the period condition and it suffices to show it holds for the cycles of a homology basis of Σ.

1.4 Examples

We furnish some examples using the Weierstrass representation of a minimal surface in its form given in Remark 2.

1. **The plane.** Let $\Sigma = \mathbb{C}$ and consider the Weierstrass data $\{g, dh\} = \{1, dz\}$. The immersion X gives a plane which is the simplest of minimal surfaces (as the Weierstrass data indicates!).

2. **The Catenoid.** Let $\Sigma = \mathbb{C} - \{0\}$ and $\{g, dh\} = \{z, \frac{dz}{z}\}$.

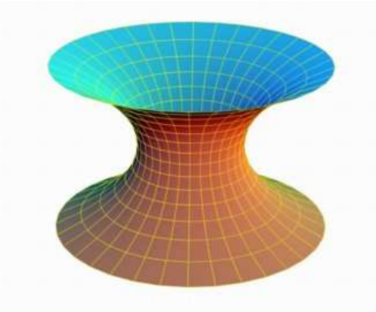


FIG. 1.1 – Catenoid

3. **The Helicoid.** Let $\Sigma = \mathbb{C}$ and $\{g, dh\} = \{e^z, i\dfrac{dz}{z}\}$.

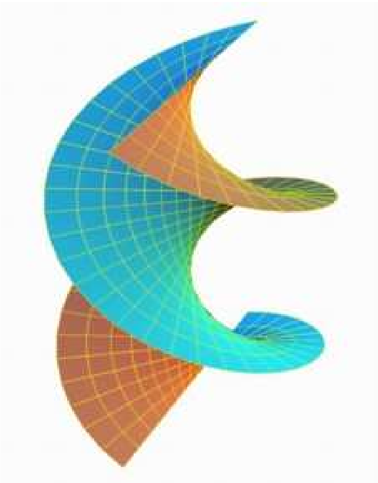


FIG. 1.2 – A fundamental piece of the Helicoid in a cylinder

4. **The Singly periodic Scherk surface.** Let $\Sigma = \mathbb{C} \cup \{\infty\}$ and $\{g, dh\} = \{z, \dfrac{1}{z^2 + z^{-2}}\dfrac{dz}{z}\}$.

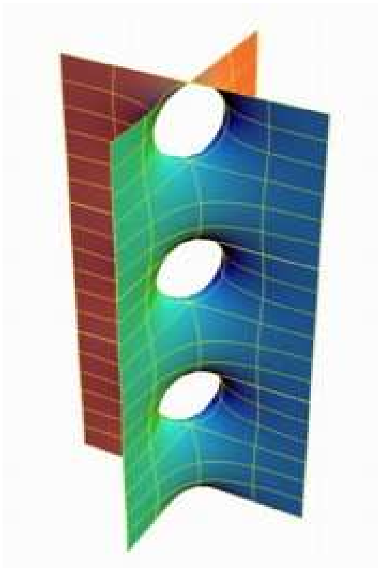


Fig. 1.3 – Singly periodic Scherk

Deuxième partie

Surfaces Minimales dans une Variété Homogène Simplement Connexe de dimension Trois et de Groupe d'Isométries de Dimension Quatre

Chapitre 2

Minimal Graphs in $\widetilde{PSL_2}(\mathbb{R})$

2.1 Introduction

In recent years there has been an increasing interest in the study of minimal and constant mean curvature surfaces in simply connected homogeneous Riemannian 3-manifolds with four dimensional isometry groups. Results in [1], like the existence of a generalized Hopf-differential or of a Schwarz reflection principle in such manifolds, suggest that these manifolds are the proper setting for studying global properties of minimal and cmc surfaces. The geometries of such manifolds have been classified by Thurston to be either those of the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, the Heisenberg group $Nil(3)$, or the fiber spaces Berger sphere and $\widetilde{PSL_2}(\mathbb{R})$(see [30]).

Certain aspects of the theory of minimal and cmc surfaces in $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and $Nil(3)$ have been studied for example in, [28], [25], [12], [7] and [1] among others.
In this paper, we study minimal graphs in $\widetilde{PSL_2}(\mathbb{R})$, known to be a Riemannian fibration over the hyperbolic plane, and we obtain a Jenkins-Serrin type theorem for such graphs over convex bounded domains in the hyperbolic plane.We emphasize that $\widetilde{PSL_2}(\mathbb{R})$ is not a product space and so one should ask what is meant by graph in such a space.

A graph in $\widetilde{PSL_2}(\mathbb{R})$ will be the image of a section of the Riemannian submersion $\pi : \widetilde{PSL_2}(\mathbb{R}) \to \mathbb{H}^2$. A Jenkins-Serrin type theorem gives necessary and sufficient conditions for the solvability of the Dirichlet problem for the minimal surface equation allowing infinite boundary values, prescribed on arcs of the boundary of a convex bounded domain in $\mathbb{H}^2$, and continuous data on the rest of the boundary. However, boundary arcs of a bounded domain in $\mathbb{H}^2$, where a solution of the minimal surface equation in $\widetilde{PSL_2}(\mathbb{R})$ admits infinite values, have to be geodesics (see section 7).

Then more precisely, let Ω be a convex bounded domain in $\mathbb{H}^2$ whose boundary consists of (open) geodesic arcs $A_1, .., A_n, B_1, ..., B_m$, together with their end points and convex open arcs $C_1, C_2, ..., C_s$. We suppose that no two geodesics A_i and no two geodesics B_i have a common end point. We give necessary and sufficient conditions for the existence of a minimal section $s : \Omega \to \widetilde{PSL_2}(\mathbb{R})$ of the Riemannian submersion, taking values $+\infty$ on the arcs $A_1, ...A_n$, $-\infty$ on the arcs $B_1, ..., B_m$ and arbitrary prescribed continuous data on the arcs $C_1, ..., C_s$.

For a simple closed geodesic polygon $\mathcal{P}$, whose vertices are chosen from among the endpoints of the segments A_i and the segments B_i, let α and β be, respectively, the total $\mathbb{H}^2$-length of the geodesics A_i and the total $\mathbb{H}^2$-length of the geodesics B_i which are part of $\mathcal{P}$. Let γ be the perimeter of $\mathcal{P}$. Note that in the case $\{C_s\} = \emptyset$, $\mathcal{P}$ could be the whole boundary of Ω.

We have the following

Theorem 4 *If the family of arcs $\{C_s\}$ is non empty, then there exists a unique section of the bundle $\pi : \widetilde{PSL_2}(\mathbb{R}) \to \mathbb{H}^2$ defined in Ω and taking the boundary values $+\infty$ on the geodesics A_i, the value $-\infty$ on the geodesics B_i and arbitrary continuous data f_s on C_s if and only if*

$$2\alpha < \gamma \text{ and } 2\beta < \gamma$$

for each polygon $\mathcal{P}$ chosen as above.

If the family of arcs $\{C_s\}$ is empty,the condition on the polygons $\mathcal{P}$ is the same except that in the case when $\mathcal{P}$ is the entire boundary of Ω then the condition is $\alpha = \beta$. Moreover, uniqueness is up to additive constants.

In $\mathbb{R}^3$ this theorem corresponds to that of Jenkins and Serrin proved in [16], and in $\mathbb{H}^2 \times \mathbb{R}$ a corresponding result was obtained in [25] by Nelli and Rosenberg. In their paper, Jenkins and Serrin make use of the a priori estimates for solutions of the minimal surface equation, proved in [31], to obtain a compactness principle for sequences of solutions and to study limit behavior of monotone sequences of solutions. They also make use of the Scherk surface as a barrier, which is fundamental to most of the results. The techniques developed by Serrin in [31] were adapted in [25] to show a Jenkins-Serrin type theorem in $\mathbb{H}^2 \times \mathbb{R}$. To obtain a priori gradient estimates for solutions of the minimal surface equation and to prove a compactness principle, we adapt a result in Spruck's [33] and we construct explicit barriers adequate to our space.

The paper is organized as follows : in section 2 we give a model for $\widetilde{PSL_2}(\mathbb{R})$, compute its metric in the coordinates and give the expression of its Levi-Civita connection. We then characterize the isometry group of $\widetilde{PSL_2}(\mathbb{R})$ based on ideas from [3] and [30]. We show that this group is generated by the lifts of isometries of $\mathbb{H}^2$ and translations along the fibers.

In section 3 we derive the minimal surface equation in $\widetilde{PSL_2}(\mathbb{R})$ and furnish examples of minimal graphs invariant under actions of one-parameter groups of isometries generated by lifts of isometries of $\mathbb{H}^2$. The rest of the paper is dedicated to develop the machinery necessary to prove our Jenkins-Serrin type theorem where we follow the main lines in [16].

In section 4 we prove an estimate for the gradient of a solution of the minimal surface equation which implies a compactness principle for sequences of solutions of the minimal surface equation in $\widetilde{PSL_2}(\mathbb{R})$ uniformly bounded on compacts of a bounded open subset of $\mathbb{H}^2$.

In section 5 we prove the existence of a solution of the Dirichlet problem for the minimal surface equation in $\widetilde{PSL_2}(\mathbb{R})$ in a convex bounded open subset of $\mathbb{H}^2$ with boundary data having possibly a finite number of discontinuities.

In sections 6 and 7 we prove a series of lemmas and propositions which will serve as machinery to prove our Jenkins-Serrin type theorem. Once this machinery is established, the lines of proof are similar to that of the corresponding Jenkins-Serrin theorem in [16] and the reader will be referred to that paper for further details.

2.2 The space $\widetilde{PSL_2}(\mathbb{R})$

The 3-dimensional Lie group of 2×2 real matrices of determinant 1 is denoted $SL_2(\mathbb{R})$. The quotient Lie group $SL_2(\mathbb{R})/\{\pm I_d\}$ is denoted $PSL_2(\mathbb{R})$ and its universal covering $\widetilde{PSL_2}(\mathbb{R})$. Of course $\widetilde{PSL_2}(\mathbb{R})$ is a Lie group itself and so admits left invariant metrics. For our purposes, it will be convenient to introduce a model for $\widetilde{PSL_2}(\mathbb{R})$ and write down explicitly the metric that interests us. In fact we shall show that $\widetilde{PSL_2}(\mathbb{R})$ is a Riemannian fibration over the hyperbolic plane, the reader can refer to [30].

Remark 4 *A homogeneous simply connected 3-manifold M with a 4-dimensional isometry group, is a Riemannian fibration over a 2-dimensional space form, and whose fibers are geodesics tangent to a unitary Killing field, say ξ. These manifolds are classified, up to isometries, by the curvature κ of the fibration base and the bundle curvature τ. The number τ is such that $\overline{\nabla}_X \xi = \tau X \times \xi$, for any vector field X ($\overline{\nabla}$ is the Levi-Civita connection of M). As we shall see in what follows, $\widetilde{PSL_2}(\mathbb{R})$ belongs to this class of manifolds and that the parameters κ and τ have the values -1 and $-\frac{1}{2}$ respectively.*

2.2.1 A model for $\widetilde{PSL_2(\mathbb{R})}$

It is known that the group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}^2$ is $PSL_2(\mathbb{R})$. Let $U\mathbb{H}^2$ denote the unit tangent bundle of $\mathbb{H}^2$, *i.e.* the submanifold of $T\mathbb{H}^2$ consisting of tangent vectors of unit length. It is easy to see that $PSL_2(\mathbb{R})$ acts transitively on $U\mathbb{H}^2$ and the stabilizer of each point under this action is trivial. This allows us to identify $PSL_2(\mathbb{R})$ and $U\mathbb{H}^2$ and consequently $\widetilde{PSL_2(\mathbb{R})}$ and $\widetilde{U\mathbb{H}^2}$.

The submanifold $U\mathbb{H}^2$ is diffeomorphically a trivial circle bundle over $\mathbb{H}^2$, meaning that $U\mathbb{H}^2 \simeq \mathbb{H}^2 \times \mathbb{S}^1$. This implies that $\widetilde{PSL_2(\mathbb{R})} \simeq \mathbb{H}^2 \times \mathbb{R}$ again from a diffeomorphic point of view.

2.2.2 Metric on $\widetilde{PSL_2(\mathbb{R})}$

A Riemannian metric on a manifold M induces a natural metric on the tangent bundle TM. We explain how this is generally done and we fix some terminology on the way, the reader can refer to [10]. Let $(p, v) \in TM$ and V a tangent vector to TM at (p, v). Choose a curve $\alpha : t \to (p(t), v(t))$ with $p(0) = p, v(0) = v$ and $V = \alpha'(0)$. Define

$$\|V\|^2_{(p,v)} = \|d\pi(V)\|^2_p + \|\frac{Dv}{dt}(0)\|^2_p,$$

where $\pi : TM \to M$ is the bundle projection and $\frac{D}{dt}$ is the covariant derivative along the curve $t \to p(t)$. The value of $\|V\|_{(p,v)}$ is independent of the choice of the curve α.

A vector at $(p, v) \in TM$ which is orthogonal to the fiber $\pi^{-1}(p) \simeq T_pM$ is said to be horizontal, and one which is tangent to the fiber is said to be vertical. We identify the vertical tangent space in $T_{(p,v)}(TM)$ to T_pM. We have

(i) $\|V\|_{(p,v)} = \|V\|_p$ if V is vertical, and,
(ii) $\|V\|_{(p,v)} = \|d\pi(V)\|_p$ if V is horizontal.

Horizontal tangent spaces have the same dimension as tangent spaces to M which implies, together with the identity (ii), that $d\pi$ induces isometries between horizontal tangent spaces and spaces tangent to M, *i.e.*,

$$d\pi : TM \to M$$

is a Riemannian submersion.

Now the metric on $\mathbb{H}^2$ induces a metric on $T\mathbb{H}^2$ which restricts to a metric on $U\mathbb{H}^2$. So we have a metric on $PSL_2(\mathbb{R})$ which lifts to a metric on its universal covering

$\widetilde{PSL_2}(\mathbb{R})$. The fact that $PSL_2(\mathbb{R})$ acts on $U\mathbb{H}^2$ by isometries implies that the metric induced on $PSL_2(\mathbb{R})$ is left invariant. This metric lifts obviously to a left invariant metric on $\widetilde{PSL_2}(\mathbb{R})$.

To see that $\widetilde{PSL_2}(\mathbb{R})$ is a Riemannian fibration over $\mathbb{H}^2$ note that the fibres of $U\mathbb{H}^2$ are 1-dimensional, hence horizontal tangent spaces to $U\mathbb{H}^2$ coincide with those of $T\mathbb{H}^2$ and π restricts to a Riemannian submersion on $U\mathbb{H}^2$. As $\widetilde{U\mathbb{H}^2}$ and $U\mathbb{H}^2$ are locally isometric we deduce that π induces a Riemannian submersion on $\widetilde{U\mathbb{H}^2}$ onto $\mathbb{H}^2$. The metric on $\widetilde{PSL_2}(\mathbb{R})$ being left invariant (hence complete) we have $\widetilde{PSL_2}(\mathbb{R})$ a complete homogeneous simply connected Riemannian manifold.

At this point we have given a model for $\widetilde{PSL_2}(\mathbb{R})$ and assigned it a metric. We next express this metric in coordinates, the reader can refer to [7].

Let $(x, y) \to \xi(x, y)$ be a conformal parametrization of $\mathbb{H}^2$ and let λ be the conformal factor so that the metric of $\mathbb{H}^2$, in these coordinates, is $\lambda^2(dx^2 + dy^2)$. As $v \in U\mathbb{H}^2$ is identified with its base point and the angle θ it makes with ∂_x we have the following local parametrization of $U\mathbb{H}^2$

$$(x, y, \theta) \to (\xi(x, y), \frac{1}{\lambda}(\cos\theta\partial_x + \sin\theta\partial_y)).$$

Let V be a tangent vector to $\widetilde{PSL_2}(\mathbb{R})$ at a point (p, v) and let $\alpha : t \to (p(t), v(t))$ be a curve passing through (p, v) at $t = 0$ and tangent to V over there. We write $p(t) = (x(t), y(t))$ and $v(t) = \dfrac{1}{\lambda}\Big(\cos\theta(t)\partial_x + \sin\theta(t)\partial_y\Big)$. Using properties of the covariant derivative along the curve $t \to p(t)$ we compute

$$\begin{aligned}\frac{Dv}{dt} =& -\frac{\lambda'}{\lambda^2}(\cos\theta\partial_x + \sin\theta\partial_y) + \frac{\theta'}{\lambda}(-\sin\theta\partial_x + \cos\theta\partial_y)\\ &+ \frac{1}{\lambda}(\cos\theta\nabla_{p'(0)}\partial_x + \sin\theta\nabla_{p'(0)}\partial_y),\end{aligned}$$

with $\lambda' = x'\lambda_x + y'\lambda_y, p'(0) = x'\partial_x + y'\partial_y, \nabla_{p'(0)}\partial_x = x'\nabla_{\partial_x}\partial_x + y'\nabla_{\partial_y}\partial_x$ and $\nabla_{p'(0)}\partial_y = x'\nabla_{\partial_x}\partial_y + y'\nabla_{\partial_y}\partial_y$. The Christoffel symbols for the metric $\lambda^2(dx^2 + dy^2)$ on $\mathbb{H}^2$ are

$$\begin{aligned}\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{\lambda_x}{\lambda}\\ -\Gamma^2_{11} = \Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = \frac{\lambda_y}{\lambda}.\end{aligned}$$

We finally obtain

$$\frac{Dv}{dt} = \frac{1}{\lambda^2}(\lambda\theta' + y'\lambda_x - x'\lambda_y)(\cos\theta\partial_y - \sin\theta\partial_x).$$

Thus

$$\|V\|^2_{(p,v)} = \lambda^2(x'^2 + y'^2) + \tfrac{1}{\lambda^2}(\lambda\theta' + y'\lambda_x - x'\lambda_y)^2$$

.

Setting $z = \theta$ on the universal covering we get the following expression for the metric on $\widetilde{PSL_2}(\mathbb{R})$:

$$ds^2 = \lambda^2(dx^2 + dy^2) + (-\frac{\lambda_y}{\lambda}dx + \frac{\lambda_x}{\lambda}dy + dz)^2.$$

Remark 5 *We can see that in our model the fibers are the vertical lines and that a unitary vector field tangent to the fibers is $\xi = \partial_z$. We can also see that translations along the fibers $(x, y, z) \to (x, y, z + a)$ are isometries generated by ξ. Thus the fibers are the trajectories of a unit Killing field and so are geodesics.*

2.2.3 An orthonormal frame on $\widetilde{PSL_2}(\mathbb{R})$

Let $\{e_1, e_2\}$ be the orthonormal frame on $\mathbb{H}^2$ with $e_1 = \lambda^{-1}\partial_x$ and $e_2 = \lambda^{-1}\partial_y$ and let E_3 be the vector field on $\widetilde{PSL_2}(\mathbb{R})$ whose expression in coordinates is ξ. Denote by E_1 and E_2 the horizontals lifts to $\widetilde{PSL_2}(\mathbb{R})$ of e_1 and e_2, *i.e.*,

$$d\pi(E_i) = e_i \text{ and } \langle E_i, E_3\rangle = 0, 1 \leqslant i \leqslant 2.$$

We remark that $d\pi(\partial x) = \partial x$ and $d\pi(\partial y) = \partial y$, then a simple computation gives the expression of E_i in coordinates,

$$E_1 = \frac{1}{\lambda}\partial_x + \frac{\lambda_y}{\lambda^2}\partial_z,\ E_2 = \frac{1}{\lambda}\partial_y - \frac{\lambda_x}{\lambda^2}\partial_z \text{ and } \mathrm{E}_3 = \partial_z.$$

In what follows let $\tilde{X}$ denote the horizontal lift to $\widetilde{PSL_2}(\mathbb{R})$ of a vector field X on $\mathbb{H}^2$; recall that $\overline{\nabla}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}[\tilde{X}, \tilde{Y}]^v$ for vector fields X, Y on $\mathbb{H}^2$. Then the Riemannian connection of $\widetilde{PSL_2}(\mathbb{R})$ is calculated in the basis $\{E_i\}$ as follows :

$$\overline{\nabla}_{E_1}E_1 = \widetilde{\nabla_{e_1}e_1} = -\frac{\lambda_y}{\lambda^2}E_2,\ \overline{\nabla}_{E_2}E_2 = \widetilde{\nabla_{e_2}e_2} = -\frac{\lambda_x}{\lambda^2}E_1.$$

As E_3 is a unitary killing field we have, for $1 \leqslant i \leqslant 3$,

$$\langle\overline{\nabla}_{E_3}E_3, E_i\rangle = -\langle\overline{\nabla}_{E_i}E_3, E_3\rangle = 0,$$

hence,

$$\overline{\nabla}_{E_3}E_3 = 0.$$

For $i, j \in \{1, 2\}$ we have,

$$\langle\overline{\nabla}_{E_j}E_i, E_j\rangle = -\langle\overline{\nabla}_{E_j}E_j, E_i\rangle \text{ and } \langle\overline{\nabla}_{E_j}E_i, E_i\rangle = 0,$$

$$2\langle\overline{\nabla}_{E_i}E_j, E_3\rangle = \langle[E_i, E_j], E_3\rangle - \langle[E_i, E_3], E_3\rangle - \langle[E_j, E_3], E_3\rangle,$$

and

$$[E_i, E_3] = 0.$$

A direct computation of $[E_1, E_2]$ gives

$$[E_1, E_2] = \frac{\lambda_y}{\lambda^2}E_1 - \frac{\lambda_x}{\lambda^2}E_2 + \Lambda E_3$$

with

$$\Lambda = \frac{\lambda_x^2 + \lambda_y^2}{\lambda^4} - \frac{\lambda_{xx} + \lambda_{yy}}{\lambda^3} = -\frac{\Delta \log \lambda}{\lambda^2}.$$

The last term of the equality is known to be the expression of the curvature, of $\mathbb{H}^2$ in this case, in terms of the conformal factor in isothermal parameters. Therefore, $\Lambda = -1$ and

$$\langle\overline{\nabla}_{E_1}E_2, E_3\rangle = -\langle\overline{\nabla}_{E_2}E_1, E_3\rangle = -\frac{1}{2}.$$

We thus obtain

$$\begin{aligned}\overline{\nabla}_{E_1}E_2 &= \frac{\lambda_y}{\lambda^2}E_1 - \frac{1}{2}E_3\\ \overline{\nabla}_{E_2}E_1 &= \frac{\lambda_x}{\lambda^2}E_2 + \frac{1}{2}E_3.\end{aligned}$$

Moreover the facts that for $1 \leqslant i \leqslant 2$,

$$\begin{gathered}[E_i, E_3] = 0,\ \langle\overline{\nabla}_{E_3}E_i, E_i\rangle = 0,\\ \langle\overline{\nabla}_{E_3}E_i, E_3\rangle = -\langle\overline{\nabla}_{E_3}E_3, E_i\rangle = 0,\\ \langle\overline{\nabla}_{E_3}E_1, E_2\rangle = \langle\overline{\nabla}_{E_1}E_3, E_2\rangle = -\langle\overline{\nabla}_{E_1}E_2, E_3\rangle = \frac{1}{2}\\ \langle\overline{\nabla}_{E_3}E_2, E_1\rangle = \langle\overline{\nabla}_{E_2}E_3, E_1\rangle = -\langle\overline{\nabla}_{E_2}E_1, E_3\rangle = -\frac{1}{2}\end{gathered}$$

conclude that

$$\begin{aligned}\overline{\nabla}_{E_3}E_1 &= \overline{\nabla}_{E_1}E_3 = \frac{1}{2}E_2,\\ \overline{\nabla}_{E_3}E_2 &= \overline{\nabla}_{E_2}E_3 = -\frac{1}{2}E_1.\end{aligned}$$

We resume our computation

$$
\begin{gathered}
\overline{\nabla}_{E_1}E_1 = -\frac{\lambda_y}{\lambda^2}E_2,\ \overline{\nabla}_{E_2}E_2 = -\frac{\lambda_x}{\lambda^2}E_1,\\
\overline{\nabla}_{E_3}E_3 = 0,\\
\overline{\nabla}_{E_1}E_2 = \frac{\lambda_y}{\lambda^2}E_1 - \frac{1}{2}E_3,\\
\overline{\nabla}_{E_2}E_1 = \frac{\lambda_x}{\lambda^2}E_2 + \frac{1}{2}E_3,\\
\overline{\nabla}_{E_3}E_1 = \overline{\nabla}_{E_1}E_3 = \frac{1}{2}E_2,\\
\overline{\nabla}_{E_3}E_2 = \overline{\nabla}_{E_2}E_3 = -\frac{1}{2}E_1.
\end{gathered}
$$

Remark 6 *The equation $\overline{\nabla}_{E_3}E_3 = 0$ is the geodesic equation for vertical fibers.*

Remark 7 *The fact that $[E_1, E_2]$ is not horizontal implies that the horizontal plane field generated by E_1 and E_2 is not integrable, meaning that there exists no horizontal surfaces in $\widetilde{PSL_2}(\mathbb{R})$.*

2.2.4 Isometries of $\widetilde{PSL_2}(\mathbb{R})$

It is known that $\widetilde{PSL_2}(\mathbb{R})$ has a four dimensional isometry group. See [30] for example, a standard reference on the geometries of 3-manifolds. However in [30] this group is characterized using Lie group theory. In what follows is what the author of this paper found a worth while simplified geometric characterization of this group based on ideas from [30] and [3].

The metric induced on the tangent bundle TM of a Riemannian manifold M is intrinsic enough that it is respected by the lifts of isometries of M to TM. In fact, each map $f \in C^\infty(M, M)$ lifts to a map $df \in C^\infty(TM, TM)$ such that $df(p, v) = (f(p), d_pf(v))$. When f is an isometry, df induces isometries on tangent spaces of TM. This can be easily seen as follows.

Let $(p, v) \in TM$ and $V \in T_{(p,v)}(TM)$ and choose a curve $\alpha(t) = (p(t), v(t))$ in TM such that $\alpha(0) = (p, v)$ and $\alpha'(0) = V$. We have,

$$
\|d_v(df)V\|^2_{(f(p),d_pf(v))} = \|d_pf(p'(0))\|^2_{f(p)} + \|\frac{Ddf(v)}{dt}(0)\|^2_{f(p)},
$$

where $\frac{D}{dt}$ is the covariant derivative along the curve $\beta(t) = df(\alpha(t))$. As d_pf is an isometry and

$$
\frac{Ddf(v)}{dt} = df(\frac{Dv}{dt})
$$

it follows directly that

$$\|d_{(p,v)}(df)V\|_{(f(p),d_pf(v))} = \|V\|_{(p,v)},$$

proving our claim.
In particular, the isometry group of $\widetilde{PSL_2}(\mathbb{R})$ contains the lifts of the isometries of $\mathbb{H}^2$. We note also that vertical translations along the fibers are isometries of $\widetilde{PSL_2}(\mathbb{R})$. These isometries read in coordinates as $(x, y, z) \to (x, y, z + a)$. So the isometry group of $\widetilde{PSL_2}(\mathbb{R})$ contains the group G generated by the lifts of isometries of $\mathbb{H}^2$ and vertical translations.

In fact, we shall show that G contains all the isometries of $\widetilde{PSL_2}(\mathbb{R})$. We begin with proving the following proposition found in [3].

Proposition 2 *The sectional curvature along a plane $P \subset T_{(p,v)}(\widetilde{PSL_2}(\mathbb{R}))$ is maximal when P contains the line $L_{(p,v)}$, the line tangent to the fiber at (p, v), and is minimal when P is orthogonal to $L_{(p,v)}$.*

Proof. Let P be a plane generated by two orthonormal vectors X and Y. Then the sectional curvature along P is given by $\langle \overline{R}(X,Y)X, Y\rangle$, where $\overline{R}$ is the curvature tensor of $\widetilde{PSL_2}(\mathbb{R})$. We have $\langle R(X,Y)X, Y\rangle = \dfrac{-7}{4} + 2(\langle X, \xi\rangle^2 + \langle Y, \xi\rangle^2)$ (see [7], proposition 2.1). As ξ is unitary we have $\langle X, \xi\rangle^2 + \langle Y, \xi\rangle^2 \leqslant 1$. So the sectional curvature will be maximal when $\langle X, \xi\rangle^2 + \langle Y, \xi\rangle^2 = 1$, and this is possible only when $\langle \xi, Z\rangle = 0$ for any vector Z such that $\{X, Y, Z\}$ forms an orthonormal basis of the tangent space to $\widetilde{PSL_2}(\mathbb{R})$ at (p, v). This means that the sectional curvature will be maximal when $\xi \in P$, *i.e.* when P contains the vertical line tangent to the fiber. Similarly we show that the sectional curvature is minimal when P is orthogonal to the vertical line tangent to the fiber □

We next show that isometries of $\widetilde{PSL_2}(\mathbb{R})$ are fiber preserving. The proposition above implies that the differential of an isometry φ sends $L_{(p,v)}$ to $L_{\varphi(p,v)}$. This follows from the fact that the differential of an isometry will send two planes along which the sectional curvature is maximal, to two planes along which the curvature is maximal. As the fiber $\pi^{-1}(p)$, tangent at the point (p, v) to $L_{(p,v)}$, is a geodesic, its image under φ is the geodesic tangent to the line $L_{\varphi(p,v)}$ at the point $\varphi(p, v)$. The fiber through $\varphi(p, v)$ is a geodesic tangent to the former line at $\varphi(p, v)$, so we conclude that it is the geodesic in question. We have then the following

Proposition 3 *The isometries of $\widetilde{PSL_2}(\mathbb{R})$ are fiber preserving, i.e. the images by an isometry of two points lying on the same fiber belong to the same fiber.*

This property will allow each isometry of $\widetilde{PSL_2}(\mathbb{R})$ to induce an isometry on $\mathbb{H}^2$ the following manner,

Lemma 1 *Every isometry φ on $\widetilde{PSL_2}(\mathbb{R})$ induces an isometry f on $\mathbb{H}^2$ such that $f \circ \pi = \pi \circ \varphi$.*

Proof. The equation $f \circ \pi = \pi \circ \varphi$ defines f the obvious way as φ is fiber preserving. For a vector $v \in T_p\mathbb{H}^2$ such that $v = d_{(p,v)}\pi(V)$, V is the horizontal lift of v, we have $d_pf(v) = d_pf(d_{(p,v)}\pi(V)) = d_{\varphi(p,v)}\pi(d_{(p,v)}\varphi(V))$. As V is horizontal and φ is an isometry we have $d_{(p,v)}\pi(V)$ also horizontal. The fact that π is a Riemannian submersion concludes that f is indeed an isometry $\square$

We proceed to show the following technical lemma found in [30], which will aid giving the finishing touch to our characterization of isometries of $\widetilde{PSL_2}(\mathbb{R})$.

Lemma 2 *Fix a point $(p, v) \in \widetilde{PSL_2}(\mathbb{R})$. We may compose any isometry α of $\widetilde{PSL_2}(\mathbb{R})$ with isometries lying in G to obtain an isometry β which fixes (p, v) and whose differential at (p, v) is the identity on the horizontal tangent plane at (p, v).*

Proof. Let f be the isometry induced by α on $\mathbb{H}^2$. We compose α with a vertical translation sending $\alpha(p, v)$ to $(f(p), d_pf(v))$ to obtain an isometry α' of $\widetilde{PSL_2}(\mathbb{R})$. Let df^{-1} denote the lift of f^{-1} to $\widetilde{PSL_2}(\mathbb{R})$ and set $\beta = df^{-1} \circ \alpha'$. This is an isometry of $\widetilde{PSL_2}(\mathbb{R})$ fixing (p, v) and leaving each horizontal vector at (p, v) invariant. In fact, for a horizontal vector V at (p, v) we have

$$d_{(p,v)}\beta(V) = d_{(f(p),d_pf(v))}df^{-1}(d_{(p,v)}\alpha(V)).$$

We denote the restriction of $d\pi$ to horizontal tangent planes by $d\pi_\circ$ and we set $w = d_p\pi(V)$, so we have

$$d_{(p,v)}\alpha(V) = d_pf(w) \text{ and } d_{(f(p),d_pf(v))}df^{-1}(d_{(p,v)}\alpha(V)) = d_p\pi_\circ^{-1}(w) = V.$$

We used the fact that $d_{(p,v)}dg(V) = d_p\pi_\circ^{-1}(d_pg(w))$, for any lift dg of an isometry g of $\mathbb{H}^2$ $\square$

At this point it is easy to prove our claim that G contains all the isometries of $\widetilde{PSL_2}(\mathbb{R})$. Let φ be an isometry and (p, v) a point of $\widetilde{PSL_2}(\mathbb{R})$. We compose φ with isometries in G and we obtain an isometry ψ which fixes (p, v), and whose differential at (p, v) is the identity on the horizontal plane at (p, v). Consequently ψ leaves invariant the fiber through (p, v) as it is fiber preserving.

Let ℓ be a piecewise geodesic loop in $\mathbb{H}^2$ based at p with non-trivial holonomy and $\tilde{\ell}$ be its horizontal lift to $\widetilde{PSL_2}(\mathbb{R})$ starting at (p, v). Let (p, w) denote the other end of $\tilde{\ell}$. Now $\psi(\tilde{\ell})$ is piecewise geodesic since so is $\tilde{\ell}$ and as ψ is an isometry (see Remark 6 below). Since ψ fixes (p, v) and the horizontal plane over there we deduce that $\psi(\tilde{\ell})$ passes through (p, v) and has the same horizontal tangent vector as $\tilde{\ell}$ there. Hence

$\psi(\tilde{\ell})$ equals $\tilde{\ell}$ and in particular ψ must fix (p, w).

As ψ is an isometry and the points (p, v) and (p, w) are distinct, due to non trivial holonomy of the geodesic loop based at p below in $\mathbb{H}^2$, it follows that ψ fixes each point of the fiber through (p, v). Then ψ is an isometry which fixes a point and whose differential over there is the identity.This implies that ψ leaves invariant geodesics through (p, v).

As our manifold is complete we can join (p, v) to any other point by a geodesic. Being an isometry ψ fixes each point of these geodesics and so ψ is the identity. This allows us to deduce that φ is a composition of elements of G.

We resume the result in the following,

Theorem 5 *The isometry group of $\widetilde{PSL_2}(\mathbb{R})$ is generated by the lifts of the isometries of $\mathbb{H}^2$ together with the vertical translations along the fibers.*

Remark 8 *Theorem 2 implies that the isometry group of $\widetilde{PSL_2}(\mathbb{R})$ is four dimensional and contains no orientation reversing isometries.*

Remark 9 *Assume that $\gamma : t \to \gamma(t)$ is a geodesic in $\mathbb{H}^2$ starting at a point p. We can lift γ to a horizontal geodesic in $\widetilde{PSL_2}(\mathbb{R})$, one whose velocity vector at each point is horizontal, starting at any point (p, v) in the fiber above p. Fix such a point (p, v) and let $v(t)$ be the parallel transport of v along γ. The curve $\bar{\gamma} : t \to (\gamma(t), v(t))$ starts at (p, v). The fact that $v(t)$ is parallel implies that $\bar{\gamma}$ is horizontal. To show that $\bar{\gamma}$ a geodesic we suppose to the contrary that it is not. We choose convex neighborhoods $W \subset \widetilde{PSL_2}(\mathbb{R})$ of (p, v) and $U \subset \mathbb{H}^2$ of p such that $\pi(W) = U$. Take two points $Q_1 = (q_1, w_1)$ and $Q_2 = (q_2, w_2)$ in $\bar{\gamma} \cap W$, joined by an arc $\bar{\alpha}$ such that $L(\bar{\alpha}) < L(\bar{\gamma}) = L(\gamma)$. In $\mathbb{H}^2$, γ is a minimizing geodesic joining q_1 and q_2. The arc $\alpha = \pi(\bar{\alpha})$ verifies $L(\alpha) \leqslant L(\bar{\alpha})$, which contradicts the fact that γ is length minimizing* (see [10], p.79).

2.3 Minimal graphs in $\widetilde{PSL_2}(\mathbb{R})$

We fix our model of $\widetilde{PSL_2}(\mathbb{R})$ as $\mathbb{H}^2 \times \mathbb{R}$ endowed with the metric

$$ds^2 = \lambda^2(dx^2 + dy^2) + \left(-\frac{\lambda_y}{\lambda}dx + \frac{\lambda_x}{\lambda}dy + dz\right)^2,$$

as described above.

We denote by $S_\circ \subset \widetilde{PSL_2}(\mathbb{R})$ the surface defined by $z = 0$. We identify a domain

$\Omega \subset \mathbb{H}^2$ and its lift to $S_\circ$. We define the graph $\Sigma(u)$ of $u \in C^0(\bar{\Omega})$ on Ω as

$$\Sigma(u) = \{(x, y, u(x, y)) \in \widetilde{PSL_2}(\mathbb{R}) | (x, y) \in \Omega\}.$$

These graphs are basically images of sections of the bundle projection

$$\pi : \widetilde{PSL_2}(\mathbb{R}) \to \mathbb{H}^2,$$

i.e. images of maps $s : \Omega \subset \mathbb{H}^2 \to \widetilde{PSL_2}(\mathbb{R})$ with $\pi \circ s = I_{\mathbb{H}^2}$. For such a map let $u(x, y)$ be the signed distance from the lift of $(x, y) \in \mathbb{H}^2$, the point of $\widetilde{PSL_2}(\mathbb{R})$ whose coordinates are $(x, y, 0)$, to $s(x, y) \in \pi^{-1}(x, y)$ along the geodesic fiber through $(x, y, 0)$. The fibers here being oriented positively by ξ. This function u defined by s defines a graph, in the sense of the above definition, which is the image of s. Clearly, each function $u \in C^0(\bar{\Omega}), \Omega \subset \mathbb{H}^2$, defines a section of the bundle projection.

For a smooth function u set $F(x, y, z) = z - u(x, y)$ so that $\Sigma(u) = F^{-1}(0)$. As F is smooth we will have

$$\eta = \frac{\overline{\nabla} F}{|\overline{\nabla} F|}$$

a unit normal field to $\Sigma(u)$.

A simple computation shows that

$$\overline{\nabla} F = (\frac{\lambda_y}{\lambda^2} - \frac{u_x}{\lambda})E_1 + (-\frac{\lambda_x}{\lambda^2} - \frac{u_y}{\lambda})E_2 + E_3.$$

Set

$$\alpha = \frac{\lambda_y}{\lambda^2} - \frac{u_x}{\lambda}, \beta = -\frac{\lambda_x}{\lambda^2} - \frac{u_y}{\lambda} \text{ and } W = |\overline{\nabla} F| = \sqrt{1 + \alpha^2 + \beta^2},$$

so that

$$\eta = \frac{\alpha}{W} E_1 + \frac{\beta}{W} E_2 + \frac{1}{W} E_3.$$

We parameterize the graph of a smooth function u by

$$(x, y) \to \phi(x, y) = (x, y, u(x, y)),$$

with $(x, y) \in \Omega$ the domain of definition of u. It is easy to see that for the metric on $\widetilde{PSL_2}(\mathbb{R})$ we have

$$\langle \phi_x, \phi_x \rangle = \lambda^2(1 + \alpha^2),\ \langle \phi_x, \phi_y \rangle = \lambda^2 \alpha\beta,\ \langle \phi_y, \phi_y \rangle = \lambda^2(1 + \beta^2),$$

giving the metric induced on the graph

$$g = \lambda^2\big((1+\alpha^2)dx^2 + \alpha\beta dxdy + \alpha\beta dydx + (1+\beta^2)dy^2\big).$$

To calculate the mean curvature H of $\Sigma(u)$, with respect to the upwards pointing normal η, choose $v_1, v_2 \in T(\widetilde{PSL_2}(\mathbb{R}))$ so that $\{v_1, v_2, \eta\}$ is an orthonormal basis of $T(\widetilde{PSL_2}(\mathbb{R}))$.

As η is a unitary field we have $\langle \overline{\nabla}_\eta \eta, \eta\rangle = 0$ and

$$\begin{aligned} 2H &= -\sum_1^2 \langle \overline{\nabla}_{v_i}\eta, v_i\rangle \\ &= -\sum_1^2 \langle \overline{\nabla}_{v_i}\eta, v_i\rangle - \langle \overline{\nabla}_\eta \eta, \eta\rangle \\ &= -\operatorname{div}(\eta). \end{aligned}$$

Therefore $2H = -\operatorname{div}\left(\dfrac{\overline{\nabla} F}{|\overline{\nabla} F|}\right)$, where div and $\overline{\nabla}$ denote respectively the divergence and the Levi-Civita connection in $\widetilde{PSL_2}(\mathbb{R})$.

Since E_1 and E_2 are the horizontal lifts of e_1 and e_2 , the facts that $\overline{\nabla}_{E_3} E_3 = 0$ and that π is a Riemannian submersion allow us to write

$$\begin{aligned} \operatorname{div}\left(\frac{\alpha}{W}E_1 + \frac{\beta}{W}E_2\right) &= \sum_1^2 \left\langle \overline{\nabla}_{E_i}\left(\frac{\alpha}{W}E_1 + \frac{\beta}{W}E_2\right), E_i\right\rangle_{\widetilde{PSL_2}(\mathbb{R})} \\ &= \sum_1^2 \left\langle \nabla_{e_i} d\pi\left(\frac{\alpha}{W}E_1 + \frac{\beta}{W}E_2\right), e_i\right\rangle_{\mathbb{H}^2} \\ &= \operatorname{div}_{\mathbb{H}^2}\Big(\frac{\alpha}{\lambda W}\partial_x + \frac{\beta}{\lambda W}\partial y\Big). \end{aligned}$$

Since E_3 is a Killing field we have $\operatorname{div}(E_3) = 0$, and

$$\operatorname{div}\Big(\frac{1}{W}E_3\Big) = \Big\langle \overline{\nabla}\Big(\frac{1}{W}\Big), E_3\Big\rangle + \frac{\operatorname{div}(E_3)}{W} = \frac{\partial}{\partial z}\Big(\frac{1}{W}\Big) = 0.$$

Therefore

$$2H = \operatorname{div}_{\mathbb{H}^2}\left(\frac{\alpha}{\lambda W}\partial_x + \frac{\beta}{\lambda W}\partial_y\right) = \operatorname{div}_{\mathbb{H}^2}\Big(d\pi(\eta)\Big).$$

We also have,

$$2H = \frac{1}{\lambda^2}\mathrm{div}_{\mathbb{R}^2}\left(\frac{\lambda\alpha}{W}\partial_x + \frac{\lambda\beta}{W}\partial_y\right),$$

as $\mathrm{div}_{\mathbb{H}^2}(X) = \dfrac{1}{\lambda^2}\mathrm{div}_{\mathbb{R}^2}(\lambda^2 X)$ for any vector field X on $\mathbb{H}^2$. The equation of a minimal graph is then

$$\mathrm{div}_{\mathbb{R}^2}\left(\frac{\lambda\alpha}{W}\partial_x + \frac{\lambda\beta}{W}\partial_y\right) = 0. \tag{2.1}$$

2.3.1 Examples of minimal surfaces and minimal graphs in $\widetilde{PSL_2}(\mathbb{R})$

In this section we find minimal graphs invariant under the action of the one parameter groups of isometries of $\widetilde{PSL_2}(\mathbb{R})$ generated by the lifts of rotations, parabolic and hyperbolic isometries of $\mathbb{H}^2$. We also determine the minimal surfaces invariant under translation along the fibers.

Example 1 Let γ be a geodesic of $\mathbb{H}^2$. The vertical cylinder over γ, $\mathcal{C}_\gamma = \pi^{-1}(\gamma) \subset \widetilde{PSL_2}(\mathbb{R})$, is a minimal surface and this can be seen as follows : Let T and η be respectively a unit tangent field and a unit normal field to γ, and let $\widetilde{T}$ and $\widetilde{\eta}$ be their corresponding horizontal lifts to $\widetilde{PSL_2}(\mathbb{R})$. We then have $\{\widetilde{T}, E_3\}$ an orthonormal basis on $\mathcal{C}_\gamma$ and $\widetilde{\eta}$ a unit normal to $\mathcal{C}_\gamma$. The mean curvature of $\mathcal{C}_\gamma$ at a point v is then

$$\begin{aligned} 2H &= -\left\langle \bar{\nabla}_{\widetilde{T}}\widetilde{\eta}, \widetilde{T}\right\rangle - \left\langle \bar{\nabla}_{E_3}\widetilde{\eta}, E_3\right\rangle - \left\langle \bar{\nabla}_{\widetilde{\eta}}\widetilde{\eta}, \widetilde{\eta}\right\rangle \\ &= \left\langle \widetilde{\nabla^{\mathbb{H}^2}_T T}, \widetilde{\eta}\right\rangle = \left\langle \nabla^{\mathbb{H}^2}_T T, \eta\right\rangle \\ &= \text{the geodesic curvature of } \gamma \text{ at the point } \pi(\text{v}), \end{aligned}$$

and as γ is a geodesic we deduce that $H = 0$, and the cylinder $\mathcal{C}_\gamma$ is thus minimal. We notice that these minimal surfaces are invariant under vertical translations and they are in fact the only ones. A minimal surface invariant under vertical translations is $\pi^{-1}(\gamma)$, where γ is a curve of $\mathbb{H}^2$. The geodesic curvature of γ is shown again by the above computation to be zero and hence γ is geodesic.

Example 2 The 1-parameter group of isometries of $\mathbb{H}^2$, given in the half plane model of $\mathbb{H}^2$ by $(x, y) \to (\epsilon x, \epsilon y)$, induces a 1-parameter group of isometries on $\widetilde{PSL_2}(\mathbb{R})$. In our model of $\widetilde{PSL_2}(\mathbb{R})$ these isometries read as $(x, y, z) \to (\epsilon x, \epsilon y, z)$. A minimal graph invariant by this group of isometries is that of a solution u of (2.1) verifying $u(r, \theta) = u(\theta)$, (r, θ) are polar coordinates on the upper half plane . Here we have

$$\lambda = \frac{1}{y},\ \alpha = -yu_x - 1 \text{ and } \beta = -yu_y,\ y > 0.$$

Let $\omega := W^2 = 1 + \alpha^2 + \beta^2$, equation (2.1) then implies

$$\omega\Big(\frac{\partial}{\partial x}(\lambda\alpha) + \frac{\partial}{\partial y}(\lambda\beta)\Big) - \frac{\lambda}{2}(\alpha\omega_x + \beta\omega_y) = 0. \tag{2.2}$$

An invariant solution u verifies

$$\begin{aligned} u_x &= \frac{\partial\theta}{\partial x}u_\theta = -\frac{\sin\theta}{r}u_\theta, \\ u_y &= \frac{\partial\theta}{\partial y}u_\theta = \frac{\cos\theta}{r}u_\theta, \\ \omega &= 2 - 2\sin^2\theta u_\theta + \sin^2\theta u_\theta^2, \\ u_{xx} &= \Big(\frac{\partial\theta}{\partial x}\Big)^2 u_{\theta\theta} + \frac{\partial^2\theta}{\partial^2 x}u_\theta = \frac{\sin^2\theta}{r^2}u_{\theta\theta} + 2\frac{\sin\theta\cos\theta}{r^2}u_\theta, \\ u_{yy} &= \Big(\frac{\partial\theta}{\partial y}\Big)^2 u_{\theta\theta} + \frac{\partial^2\theta}{\partial^2 y}u_\theta = \frac{\cos^2\theta}{r^2}u_{\theta\theta} - 2\frac{\sin\theta\cos\theta}{r^2}u_\theta. \end{aligned}$$

Equation (2.2) implies that

$$\omega u_{\theta\theta} - \frac{1}{2}\omega_\theta(u_\theta - 1) = 0 \tag{2.3}$$

from which we deduce that either
(i) $u_\theta = 1$, or
(ii)

$$2\frac{u_{\theta\theta}}{u_\theta - 1} = \frac{\omega_\theta}{\omega}$$

which is equivalent to

$$\frac{(u_\theta - 1)^2}{\omega} = C,\ C \geq 0.$$

The cases (i) and (ii) are resumed in

$$(1 - C\sin^2\theta)(u_\theta^2 - 2u_\theta) = 2C - 1, C \geq 0.$$

For $0 \leqslant C < 1$, this first integral defines a 1-parameter family of graphs over the hyperbolic plane, given up to an additive constant by

$$u(r,\theta) = u(\theta) = \pm\sqrt{C}\int_0^\theta \frac{\sqrt{1+\cos^2\theta}}{\sqrt{1 - C\sin^2\theta}}d\theta + \theta,\ 0 < \theta < \pi.$$

For example, when $C = 0$ we obtain up to vertical translations, half a (euclidean) Helicoid over the hyperbolic plane.

When $C = \dfrac{1}{2}$ the above solutions simplify to $u(r,\theta) = \theta \pm \theta + \mathit{constant}$. So on the one hand we obtain up to vertical translations, half a Helicoid stretched in the vertical direction. It is the surface over the hyperbolic plane obtained by rotating, in euclidean terms, the $x-$axis about the $z-$axis, and translating it vertically twice as fast. On the other hand we obtain translates of the plane $\{z = 0\}$ as invariant minimal surfaces which correspond to the solutions $u(r, \theta) = constant$.

For $C = 1$, we obtain solutions defined in the first and the second quadrants of the hyperbolic plane. The solutions are

$$u(r,\theta) = u(\theta) = \int_0^\theta \frac{\sqrt{1+\cos^2\theta}}{\cos\theta} d\theta + \theta$$
$$\Big(= -\int_0^\theta \frac{\sqrt{1+\cos^2\theta}}{\cos\theta} d\theta + \theta \text{ respectively}\Big),\ 0 < \theta < \frac{\pi}{2},$$

defined in the first quadrant and taking values 0 on the positive $x-$axis and $+\infty$ ($-\infty$ respectively) on the $y-$axis. On the other hand the solutions

$$u(r,\theta) = u(\theta) = \int_{\frac{\pi}{2}}^\theta \frac{\sqrt{1+\cos^2\theta}}{\cos\theta} d\theta + \theta$$
$$\Big(= -\int_{\frac{\pi}{2}}^\theta \frac{\sqrt{1+\cos^2\theta}}{\cos\theta} d\theta + \theta, \text{ respectively}\Big), \frac{\pi}{2} < \theta < \pi,$$

defined in the second quadrant and taking values $+\infty$ ($-\infty$ respectively) on the $y-$axis and 0 on the negative $x-$axis. The solutions obtained so far define complete minimal graphs.

For $C > 1$, the equation $1 - C\sin^2\theta = 0$ has two solutions, say θ_1 and $\theta_2 = \pi - \theta_1$, in $]0, \pi[$ such that $\theta_1 < \frac{\pi}{2} < \theta_2$. The first integral defines a one-parameter family of disconnected graphs defined in the region $\{0 < \theta < \theta_1\} \bigcup \{\theta_2 < \theta < \pi\}$. We have, up to additive constants, the solutions

$$u(r,\theta) = u(\theta) = \sqrt{C}\int_0^\theta \frac{\sqrt{1+\cos^2\theta}}{\sqrt{1-C\sin^2\theta}} d\theta + \theta$$
$$\Big(= -\sqrt{C}\int_0^\theta \frac{\sqrt{1+\cos^2\theta}}{\sqrt{1-C\sin^2\theta}} d\theta + \theta \text{ respectively}\Big),\ 0 < \theta < \theta_1,$$

and

$$u(r,\theta) = u(\theta) = \sqrt{C}\int_\theta^\pi \frac{\sqrt{1+\cos^2\theta}}{\sqrt{1-C\sin^2\theta}} d\theta + \theta$$
$$\Big(= -\sqrt{C}\int_\theta^\pi \frac{\sqrt{1+\cos^2\theta}}{\sqrt{1-C\sin^2\theta}} d\theta + \theta \text{ respectively}\Big),\ \theta_2 < \theta < \pi.$$

One can see easily that the solutions have finite values over the lines $\theta = \theta_1$ and $\theta = \theta_2$ and admit vertical tangent planes over there. However, the solutions obtained for these values of the parameter C do not define complete minimal graphs.

One obtains complete minimal surfaces above the region $\{0 < \theta \leq \theta_1\}$ for example, when one considers unions of graphs $u(r, \theta)$ above that region. We consider the graphs obtained for both factors $\pm\sqrt{C}$ of the integral in the above expression of u and translate them vertically to take values θ_1 over $\theta = \theta_1$.To see that the union defines a regular surface above $\theta = \theta_1$ we simply show that θ is a smooth function of z near $z = \theta_1$.

We have $z = u(\theta)$ which implies that the derivatives of θ with respect to z are given by

$$\frac{\partial u}{\partial \theta} = \frac{1}{\theta'} \text{ and } \frac{\partial^2 u}{\partial \theta^2} = -\frac{\theta''}{\theta'^3}.$$

We compute ω and ω_θ in terms of θ and its derivatives then substitute in (2.3) to obtain after necessary simplifications,

$$\theta''(2 - \sin^2 \theta) + \sin\theta\cos\theta(\theta' - 1)(2\theta' - 1) = 0. \tag{2.4}$$

As the graphs $u(r, \theta)$ admit vertical tangent planes at the points $z = \theta_1$, θ defines a C^1-function of z and the equation (2.4) shows then that θ is in fact smooth. $\square$

Example 3 Consider the disc model for the hyperbolic plane. Rotations, in euclidean terms, about the center of the disc are isometries of $\mathbb{H}^2$. The lifts of these isometries to $\widetilde{PSL_2}(\mathbb{R})$, seen in our model, are euclidean screw motions. The image of a point (x, y, z) is obtained by rotating the (x, y) part around the $z-$axis then translating it along the $z-$axis by the same amount. We can then compose the lift of a rotation on $\mathbb{H}^2$ with a translation along a vertical fiber to obtain an isometry of $\widetilde{PSL_2}(\mathbb{R})$ which is rotation about the fiber. So we have a 1-parameter group of isometries of $\widetilde{PSL_2}(\mathbb{R})$ which are, in our model, rotations about the $z-$axis.

A minimal graph invariant by this group is that of a solution u of (2.2) verifying $u(r, \theta) = u(r)$, (r, θ) polar coordinates on the disc.

Here we have,

$$\lambda = \frac{1}{1 - \frac{x^2+y^2}{4}}, \alpha = -\frac{u_x}{\lambda} + \frac{y}{2} \text{ and } \beta = -\frac{u_y}{\lambda} - \frac{x}{2}, \ x^2 + y^2 < 4.$$

An invariant solution verifies

$$\begin{aligned} u_x &= \frac{\partial r}{\partial x} u_r = \frac{x}{r} u_r, \\ u_y &= \frac{\partial r}{\partial y} u_r = \frac{y}{r} u_r, \end{aligned}$$

$$\omega = 1 + \frac{r^2}{4} + \frac{1}{\lambda^2}u_r^2,$$
$$u_{xx} = \left(\frac{\partial r}{\partial x}\right)^2 u_{rr} + \frac{\partial^2 r}{\partial^2 x}u_r = \frac{x^2}{r^2}u_{rr} + \frac{y^2}{r^3}u_r,$$
$$u_{yy} = \left(\frac{\partial r}{\partial y}\right)^2 u_{rr} + \frac{\partial^2 r}{\partial^2 y}u_r = \frac{y^2}{r^2}u_{rr} + \frac{x^2}{r^2}u_r.$$

Equation (2.2) implies that

$$\omega(u_{rr} + \frac{1}{r}u_r) - \frac{1}{2}u_r\omega_r = 0,$$

from which we deduce that either

(i) $u \equiv constant$
or
(ii)

$$2\frac{u_{rr}}{u_r} + \frac{2}{r} = \frac{\omega_r}{\omega}$$

which is equivalent to

$$r^2u_r^2 = C\omega,\ C > 0.$$

This implies that

$$u_r = \pm 2\sqrt{\frac{r^2+4}{Cr^2 - (r^2-4)^2}},$$

with $C > 0$ and $0 < r_\circ < r < 2$, $r_\circ = \sqrt{\dfrac{8 + C - \sqrt{(8+C)^2 - 64}}{2}}$.

Remark that $u_r(r_\circ) = \pm\infty$ and that the solutions are either increasing or decreasing in r. In a fashion similar to that in the above example, we show that the union of the graphs corresponding to both values of u_r and taking the value 0 at $r_\circ$ define a regular surface. Then this first integral defines up to vertical translations, a family of minimal surfaces of catenary type. As C varies in $]0, +\infty[$ the asymptotic angles at infinity between the members of the family and the cylinder $\partial\mathbb{H}^2 \times \mathbb{R}$ assume all the values in $]0, \pi[$.

Remark also that up to a vertical translation, when $C \to +\infty$, $r_\circ \to 0$ and the limit surface is the doubly covered hyperbolic plane(identified with $z = 0$). When $C \to 0$, $r_\circ \to 2$ and up to a vertical translation the family degenerates to the circle at infinity $\partial\mathbb{H}^2$ (doubly covered).

Example 4 The 1-parameter group of isometries of $\mathbb{H}^2$, given in the half plane model of $\mathbb{H}^2$ by $(x,y) \to (x+a,y)$, induces a 1-parameter group of isometries on $\widetilde{PSL_2}(\mathbb{R})$. In our model of $\widetilde{PSL_2}(\mathbb{R})$ these isometries read as $(x,y,z) \to (x+a,y,z)$.

A minimal graph invariant by this group of isometries is that of a solution u of (2.1) verifying $u(x,y) = u(y),\ y > 0$.

We have

$$\lambda = \frac{1}{y},\ \alpha = -1,\ \beta = -yu_y \text{ and } \omega = 2 + y^2 u_y^2,$$

and so equation (2.2) implies that

$$\omega u_{yy} - \frac{1}{2} u_y \omega_y = 0.$$

We deduce that either $u \equiv$ *constant*, or $u_y = \pm \frac{\sqrt{2}}{\sqrt{C^2 - y^2}},\ C > 0$. This equation defines up to additive constants, surfaces symmetric (in euclidean terms) with respect to $\{z = 0\}$. These surfaces are the union of the two graphs

$$u(x,y) = \pm\sqrt{2} \arcsin\left(\frac{y}{C}\right) \mp \frac{\sqrt{2}\pi}{2}$$

over the region $\{0 < y \leq C\}$. As $C \to +\infty$ the limit surface is $\{z = 0\}$ (doubly covered).

$\square$

Remark 10 *There exists no compact complete minimal surface in $\widetilde{PSL_2}(\mathbb{R})$. For otherwise, if such a surface Σ exists, we may then translate down any minimal surface $z =$ constant not intersecting Σ until there is a first contact point. This implies that the two surfaces are tangent and one above the other. By the maximum principle Σ will be equal to a surface $z =$ constant. This is a contradiction as the surfaces $z =$ constant are not compact.*

Remark 11 *There exist no complete proper minimal surfaces in $\widetilde{PSL_2}(\mathbb{R})$ with bounded projection into $\mathbb{H}^2$. For assume that such a surface Σ exist and remark that we may choose values of C for which the surfaces of example 1 either intersect or are disjoint with Σ. This means that there's a value of C for which there's a first point of contact between Σ and the corresponding example. This point cannot be at ∞ as can be easily seen. Hence the two surfaces will be tangent which is impossible by the maximum principle.*

2.4 Gradient Estimates

We will next prove an estimate for the gradient of a solution $u : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$ of (2.1), which will be fundamental for proving later results, following the lines of [33]. For this aim we will need the following formulae which hold for surfaces in 3-manifolds and in particular for surfaces $\Sigma \subset \widetilde{PSL_2(\mathbb{R})}$:

$$\begin{aligned} |\nabla_\Sigma f|^2 &= |\overline{\nabla}\tilde{f}|^2 - \langle \overline{\nabla}\tilde{f}, \eta\rangle^2 && (2.5)\\ \Delta_\Sigma f &= 2\langle \overline{\nabla}\tilde{f}, \eta\rangle H + \Delta\tilde{f} - \langle \overline{\nabla}_\eta \overline{\nabla}\tilde{f}, \eta\rangle && (2.6)\\ \Delta_\Sigma g(f) &= g'(f)\Delta_\Sigma f + g''(f)|\nabla_\Sigma f|^2, && (2.7)\end{aligned}$$

where f is a function defined on Σ, or the restriction to Σ of a function

$$\tilde{f} : \widetilde{PSL_2(\mathbb{R})} \to \mathbb{R}^2,$$

H is the mean curvature of Σ and η a unit normal field on Σ. We will also need the following fact, if $X : M \to N$ is a constant mean curvature isometric immersion of a surface M in a 3-manifold N, and if η is a unit normal field to M and ξ a Killing field on N then the function $n = \langle \eta, \xi\rangle$ verifies the following equation

$$\Delta_\Sigma n = -(|A|^2 + Ric(\eta))n, \tag{2.8}$$

where $|A|$ is the norm of the second fundamental form of M and Ric is the Ricci curvature of N.
For minimal graphs in $\widetilde{PSL_2(\mathbb{R})}$, $n = \dfrac{1}{W}$, so that

$$\Delta_\Sigma \frac{1}{W} = -(|A|^2 + Ric(\eta))\frac{1}{W} \tag{2.9}$$

with

$$Ric(\eta) = -\frac{3}{2} + \frac{2}{W^2},$$

which we compute using the equations of proposition 2.1 in [7].

Remark 12 *Equation* (2.9) *implies that minimal graphs in* $\widetilde{PSL_2(\mathbb{R})}$ *are stable. This follows directly from the definition of stability of a minimal surface and Theorem 1 in [13].*

We finally note that a function $\phi : \Omega \to \mathbb{R}$ lifts as a section of π to a function on $\widetilde{PSL_2(\mathbb{R})}$, whose restriction to Σ will be also denoted by ϕ. Then using (2.5) we obtain

$$|\nabla_\Sigma \phi|^2_\Sigma = \frac{1}{\lambda^2 W^2}\big((\phi_x^2 + \phi_y^2) + (\beta\phi_x - \alpha\phi_y)^2\big)$$

which implies that

$$|\nabla_\Sigma \phi|^2_\Sigma \geqslant \frac{1}{W^2}|D\phi|^2_{\mathbb{H}^2}. \tag{2.10}$$

Theorem 6 *Let u be a non-negative solution of the minimal surface equation* (2.1) *in a bounded domain $\Omega \subset \mathbb{H}^2$. Then at each point $p \in \Omega$ we have*

$$W(p) \leqslant C$$

where C is a positive constant which depends only on $u(p)$, the distance of p to $\partial\Omega$ and on bounds of λ and its derivatives on Ω.

Proof. We fix a point $p \in \Omega$. We introduce the function $f = \mu(x)W$ on a geodesic ball $B_\rho(p) \subset \Omega \subset \mathbb{H}^2$, for which we will derive a maximum principle by computing $\Delta_\Sigma f$. The function μ is to be defined. We have

$$\begin{aligned}\Delta_\Sigma f &= W\Delta_\Sigma \mu + 2\langle \nabla_\Sigma W, \nabla_\Sigma \mu\rangle + \mu\Delta_\Sigma W \\ &= W\Delta_\Sigma \mu + \frac{2}{W}\big(\langle \nabla_\Sigma W, \nabla_\Sigma f\rangle - \mu|\nabla_\Sigma W|^2\big) + \mu\Delta_\Sigma W.\end{aligned}$$

We then obtain

$$\Delta_\Sigma f - \frac{2}{W}\langle \nabla_\Sigma W, \nabla_\Sigma f\rangle = \mu\big(\Delta_\Sigma W - \frac{2}{W}|\nabla_\Sigma W|^2\big) + W\Delta_\Sigma \mu.$$

However from (2.7) we get

$$\Delta_\Sigma \frac{1}{W} = -\frac{1}{W^2}\Delta_\Sigma W + \frac{2}{W^3}|\nabla_\Sigma W|^2,$$

and (2.8) then implies that

$$\Delta_\Sigma W - \frac{2}{W}|\nabla_\Sigma W|^2 = (|A|^2 + Ric(\eta))W,$$

so that

$$\Delta_\Sigma W - \frac{2}{W}|\nabla_\Sigma W|^2 \geqslant Ric(\eta)W \geqslant -\frac{3}{2}W$$

We get

$$\Delta_\Sigma f - \frac{2}{W}\langle \nabla_\Sigma W, \nabla_\Sigma f\rangle \geqslant W\big(\Delta_\Sigma \mu - \frac{3}{2}\mu\big).$$

The idea is to define μ so that $\Delta_\Sigma \mu - \dfrac{3}{2}\mu > 0$.
We set

$$\mu(x) = e^{K\phi} - 1, \text{ and } \phi(x) = -\frac{u(x)}{2u_\circ} + 1 - \Big(\frac{d(x)}{\rho}\Big)^2$$

on the ball $B(p,\rho)$, where $u_\circ = u(p)$, d is the geodesic distance from p and $K > 0$ a constant to be determined. We next bound $\Delta_\Sigma \mu - \frac{3}{2}\mu$ from below. Using (2.7) we obtain

$$\Delta_\Sigma \mu = K e^{K\phi} \Delta_\Sigma \phi + K^2 e^{K\phi} |\nabla_\Sigma \phi|^2.$$

As $u = h_{|\Sigma}$, $h = z$ in the given model of $\widetilde{PSL_2}(\mathbb{R})$ and Σ minimal, (2.6) implies that

$$\Delta_\Sigma \ u = \Delta h - \langle \overline{\nabla}_\eta \overline{\nabla} h, \eta \rangle,$$

showing that we can bound $\Delta_\Sigma u$ by a constant independent of u. Similarly we bound $\Delta_\Sigma d^2$ which shows that

$$\Delta_\Sigma \phi \geqslant -C_1 \big(\frac{1}{u_\circ} + \frac{1}{\rho^2} \big),$$

where C_1 is a constant. The inequality (2.10) implies that in $B_\rho(p)$

$$|\nabla_\Sigma \phi|^2_\Sigma \geqslant \frac{1}{W^2} |D\phi|^2_{\mathbb{H}^2} \geqslant \frac{1}{W^2} \Big(\frac{|Du|^2_{\mathbb{H}^2}}{4u_\circ^2} - \frac{2}{u_\circ \rho} |Du|_{\mathbb{H}^2} \Big),$$

which implies that when

$$|Du|_{\mathbb{H}^2} \geqslant \frac{16 u_\circ}{\rho}$$

we have

$$|\nabla_\Sigma \phi|^2_\Sigma \geqslant \frac{|Du|^2_{\mathbb{H}^2}}{8u_\circ^2 W^2}.$$

Now as

$$W^2 \leqslant 1 + 2|Du|^2_{\mathbb{H}^2} + 2\Big(\big(\frac{\lambda_x}{\lambda^2}\big)^2 + \big(\frac{\lambda_y}{\lambda^2}\big)^2 \Big) \tag{2.11}$$

we obtain

$$\frac{|Du|^2_{\mathbb{H}^2}}{W^2} \geqslant C_2 \frac{|Du|^2_{\mathbb{H}^2}}{1 + |Du|^2_{\mathbb{H}^2}},$$

where C_2 is a positive constant which depends only on bounds of λ and its derivatives over Ω. Hence on the set where $|Du|_{\mathbb{H}^2} > max(1, \dfrac{16u_\circ}{\rho})$ we find

$$\Delta_\Sigma \mu - \frac{3}{2}\mu \geqslant C' e^{K\phi} \Big(\frac{C}{u_\circ^2} K^2 - \big(\frac{1}{u_\circ} + \frac{1}{\rho^2} \big) K - 1 \Big),$$

where C and C' are positive constants which depend on bounds of λ and its derivatives. We next choose

$$K > \frac{u_\circ^2}{2C}\Big(\frac{1}{u_\circ} + \frac{1}{\rho^2} + \sqrt{\big(\frac{1}{u_\circ} + \frac{1}{\rho^2}\big)^2 + \frac{4C}{u_\circ^2}}\Big)$$

so that $\Delta_\Sigma \mu - \dfrac{3}{2}\mu > 0$ on the set $|Du|_{\mathbb{H}^2} > max(1, \dfrac{16u_\circ}{\rho})$.
If

$$|Du|_{\mathbb{H}^2} \leq max(1, \tfrac{16u_\circ}{\rho})$$

then inequality (2.11) proves our claim on $W(p)$. Otherwise, we consider the open set

$$U = \{x \in B_\rho(p)/\phi > 0, |Du|_{\mathbb{H}^2} > max(1, \frac{16u_\circ}{\rho})\},$$

and note that $p \in U$.Then by the maximum principle, the point $p_\circ$ where f achieves its maximum on U belongs to ∂U with $f(p_\circ) > 0$. As $\phi < 0$ on $\partial B_\rho(p)$ we have

$$\partial U \cap \partial B_\rho(p) = \emptyset$$

and therefore

$$p_\circ \in \{|Du|_{\mathbb{H}^2} = max(1, \frac{16u_\circ}{\rho})\} \cap \{\phi > 0\}.$$

Therefore

$$f(p) = \mu(p)W(p) \leqslant \mathcal{C}\mu(p_\circ)\sqrt{1 + max^2(1, \frac{16u_\circ}{\rho})}$$

and

$$W(p) \leqslant \mathcal{C}e^{\frac{K}{2}}\sqrt{1 + max^2(1, \frac{16u_\circ}{\rho})},$$

where $\mathcal{C}$ is a positive constant which depends only on bounds of λ and its derivatives. The proof is completed. □

Corollary 1 *Let u be a bounded solution of the minimal surface equation* (2.1) *in a domain $\Omega \subset \mathbb{H}^2$. Then at any point $p \in \Omega$ we have*

$$W(p) \leq \mathcal{C}$$

where $\mathcal{C}$ is a positive constant which depends only on $\max_{\partial\Omega} |u|$, the distance of p to $\partial\Omega$ and on bounds of λ and its derivatives on Ω.

Theorem 7 *Let u be a solution of the minimal surface equation* (2.1) *in Ω with $W \leqslant M$ at a point $p \in \Omega$. Then there exists R, which depends only on $M, u(p)$ and $d(p, \partial\Omega)$, such that $W \leqslant 2M$ on $D(p,R)$.*

Proof. We shall derive an estimate on $\|\nabla W\|$, the norm of the $\mathbb{R}^2$-gradient of W, from which the bound on W follows readily. The graph of u is parametrized by

$$(x,y) \longrightarrow \psi(x,y) = (x,y,u(x,y)),$$

and a unit normal field to the graph is

$$\eta = \frac{\alpha}{W}E_1 + \frac{\beta}{W}E_2 + \frac{1}{W}E_3.$$

The partial derivatives of ψ,

$$\psi_x = \partial_x + u_x\partial_z = \lambda E_1 - \lambda\alpha E_3$$

and

$$\psi_y = \partial_y + u_x\partial_z = \lambda E_2 - \lambda\beta E_3,$$

are such that

$$\|\psi_x\|^2_{\widetilde{PSL_2(\mathbb{R})}} \leqslant \lambda W \text{ and } \|\psi_y\|^2_{\widetilde{PSL_2(\mathbb{R})}} \leqslant \lambda W.$$

We shall estimate the partial derivatives of α and β by applying the Schoen curvature estimate. For this purpose we need to calculate $\|\overline{\nabla}_{\psi_x}\eta\|$,

$$\begin{aligned}\overline{\nabla}_{\psi_x}\eta =& \frac{\partial}{\partial x}\Big(\frac{\alpha}{W}\Big)E_1 + \frac{\alpha}{W}(\lambda\overline{\nabla}_{E_1}E_1 - \lambda\alpha\overline{\nabla}_{E_3}E_1)\\ &+ \frac{\partial}{\partial x}\Big(\frac{\beta}{W}\Big)E_2 + \frac{\beta}{W}(\lambda\overline{\nabla}_{E_1}E_2 - \lambda\alpha\overline{\nabla}_{E_3}E_2)\\ &+ \frac{\partial}{\partial x}\Big(\frac{1}{W}\Big)E_3 + \frac{1}{W}(\lambda\overline{\nabla}_{E_1}E_3 - \lambda\alpha\overline{\nabla}_{E_3}E_3),\end{aligned}$$

so that

$$\overline{\nabla}_{\psi_x}\eta = U + V$$

with

$$\begin{aligned}U =& \Big(\frac{\partial}{\partial x}\Big(\frac{\alpha}{W}\Big) + \frac{\lambda_y}{\lambda}\frac{\beta}{W}\Big)E_1\\ &+ \Big(\frac{\partial}{\partial x}\Big(\frac{\beta}{W}\Big) - \frac{\lambda_y}{\lambda}\frac{\alpha}{W}\Big)E_2\\ &+ \frac{\partial}{\partial x}\Big(\frac{1}{W}\Big)E_3.\end{aligned}$$

and

$$V = \frac{\lambda\alpha\beta}{2W}E_1 + \frac{\lambda(1-\alpha^2)}{2W}E_2 - \frac{\lambda\beta}{2W}E_3.$$

It is easy to see that

$$\|U\|^2_{\widetilde{PSL_2(\mathbb{R})}} \leqslant 2\Big(\|\overline{\nabla}_{\psi_x}\eta\|^2_{\widetilde{PSL_2(\mathbb{R})}} + \|V\|^2_{\widetilde{PSL_2(\mathbb{R})}}\Big).$$

We wish to estimate $\|U\|_{\widetilde{PSL_2(\mathbb{R})}}$ as it is the term which contains derivatives of α and β. We have

$$\begin{aligned}
\frac{\partial}{\partial x}\Big(\frac{\alpha}{W}\Big) &= \frac{1}{W^3}\Big((1+\beta^2)\alpha_x - \alpha\beta\beta_x\Big)\\
\frac{\partial}{\partial x}\Big(\frac{\beta}{W}\Big) &= \frac{1}{W^3}\Big((1+\alpha^2)\beta_x - \alpha\beta\alpha_x\Big)\\
\frac{\partial}{\partial x}\Big(\frac{1}{W}\Big) &= -\frac{\alpha\alpha_x + \beta\beta_x}{W^3}.
\end{aligned}$$

Therefore,

$$\|U\|^2_{\widetilde{PSL_2(\mathbb{R})}} = \frac{1}{W^4}(\alpha_x^2 + \beta_x^2) + \Big(\frac{1}{W^2}(\alpha_x\beta - \alpha\beta_x) + \frac{\lambda_y}{\lambda}\Big)^2 - \Big(\frac{\lambda_y}{\lambda}\Big)^2\frac{1}{W^2}.$$

Its easy to see that

$$\|V\|_{\widetilde{PSL_2(\mathbb{R})}} \leqslant \lambda W.$$

The shape operator of the graph, which is stable (c.f. Remark 2.9, is $\widetilde{A}\psi_x = -\overline{\nabla}_{\psi_x}\eta$ and the Schoen curvature estimate implies that $|\widetilde{A}| \leqslant C$ in a disc about each point on the graph, a constant which depends only on the $\widetilde{PSL_2(\mathbb{R})}$ distance of the point from the boundary of the graph. The inequality

$$\|\widetilde{A}\psi_x\|_{\widetilde{PSL_2(\mathbb{R})}} \leqslant |\widetilde{A}|\|\psi_x\|_{\widetilde{PSL_2(\mathbb{R})}},$$

implies that at each point $p \in \Omega$

$$\alpha_x^2 + \beta_x^2 \leqslant \lambda^2 C W^6 + \lambda^2 W^6 + \Big(\frac{\lambda_y}{\lambda}\Big)^2 W^2,$$

and yet

$$\alpha_x^2 + \beta_x^2 \leqslant C W^6,$$

C is a constant which depends only on $u(p)$, the distance of p from $\partial\Omega$ and on bounds of λ and its derivatives over compacts of Ω.
Similarly we obtain

$$\overline{\nabla}_{\psi_y}\eta = U' + V'$$

with

$$\begin{aligned} U' =&\Big(\frac{\partial}{\partial y}\Big(\frac{\alpha}{W}\Big) - \frac{\lambda_x}{\lambda}\frac{\beta}{W}\Big)E_1 \\ &+ \Big(\frac{\partial}{\partial y}\Big(\frac{\beta}{W}\Big) + \frac{\lambda_x}{\lambda}\frac{\alpha}{W}\Big)E_2 \\ &+ \frac{\partial}{\partial y}\Big(\frac{1}{W}\Big)E_3, \end{aligned}$$

and

$$V' = \frac{\lambda(\beta^2-1)}{2W}E_1 - \frac{\lambda\alpha\beta}{2W}E_2 + \frac{\lambda\alpha}{2W}E_3$$

The facts

$$\|U'\|^2_{\widetilde{PSL_2(\mathbb{R})}} = \frac{1}{W^4}(\alpha_y^2+\beta_y^2) + \Big(\frac{1}{W^2}(\alpha\beta_y - \beta\alpha_y) + \frac{\lambda_x}{\lambda}\Big)^2 - \Big(\frac{\lambda_x}{\lambda}\Big)^2\frac{1}{W^2},$$

and

$$\|V'\| \le \lambda W$$

imply that

$$\alpha_y^2 + \beta_y^2 \leqslant CW^6,$$

C is a constant which depends only on Ω, $u(p)$ and the distance of p from the boundary of Ω.
Note that $\nabla W = \dfrac{1}{W}(\alpha\alpha_x + \beta\beta_x, \alpha\alpha_y + \beta\beta_y)$, hence the estimates obtained on the partial derivatives of α and β imply that at each point $p \in \Omega$,

$$\|\nabla W\| \leqslant CW^3.$$

This estimate will allow us to conclude our proof. Let $R = \frac{1}{2}d_{\mathbb{R}^2}(p, \partial\Omega)$ and introduce the function $f(r) = W(r,\theta)$ in $D(p,R) \subset \Omega$, where r and θ are the polar coordinates with origin p. We fix $\theta \neq 0$ and we remark that $f(0) = W(p) \le M$ and

$$f'(r) = \frac{\partial W}{\partial r} \leqslant \|\nabla W\| \leqslant Cf(r)^3.$$

Integrating this inequality we obtain that $f(r) \leqslant 2M$ for $r \in [0, \frac{3}{8M^2C}[$, which reads into W is bounded by $2M$ on $D(p, min(R, \frac{3}{8M^2C}))$. $\square$

The above estimates imply that the first and second derivatives of a solution u at a point p, admit bounds which depend only on the value of u at p, the distance of p from the boundary and on Ω. Then the classical Ascoli theorem implies the following

Compactness principle. Let (u_n) be a uniformly bounded sequence of solutions of the minimal surface equation (1)in a domain Ω. Then there exists a subsequence which converges to a solution in Ω, the convergence being uniform on every compact subset of Ω.

2.5 Preliminary Existence Theorems

In what follows C will denote a rectifiable Jordan curve in $\widetilde{PSL_2}(\mathbb{R})$. Let $\mathcal{D}$ denote the solution of the Plateau problem for C (exists as $\widetilde{PSL_2}(\mathbb{R})$ is homogeneous, see [24]), a compact minimal disc with least area, having C as boundary. It is known that $\mathcal{D}$ has a tangent plane at each interior point, see [19]. Let h denote the function defined on $\widetilde{PSL_2}(\mathbb{R})$ whose expression in the model described above is $h = z$ and set $m_C = \min\limits_C(h)$ and $M_C = \max\limits_C(h)$. We suppose that $m < M$ for our curve C for otherwise $\mathcal{D}$ will be a piece of a surface defined by $h = constant$.
For a curve $\gamma \subset \mathbb{H}^2$, we denote $\mathcal{C}(\gamma)$ the convex hull of γ, *i.e.* the smallest (geodesically) convex subset of $\mathbb{H}^2$ containing γ and $R_C = \pi^{-1}(\mathcal{C}(\pi(C)))$, the region in $\widetilde{PSL_2}(\mathbb{R})$ above the convex hull of the projection of C. Note that C is contained in R_C. The following proposition corresponds in $\mathbb{R}^3$ to the result that a minimal surface is contained in the convex hull of its boundary.

Proposition 4 *The minimal disc $\mathcal{D}$ is contained in $R_C \bigcap \{m_C \leqslant h \leqslant M_C\}$.*

Proof. There exists a minimal disc Δ defined by $h = constant$ not intersecting $\mathcal{D}$. If $\mathcal{D}$ had an interior point p above (respectively below) all other points of C, we would translate Δ downwards (respectively upwards) along vertical fibers so that Δ is eventually tangent to $\mathcal{D}$. This is impossible by the maximum principle as we assume h non-constant on C. Therefore $\mathcal{D}$ is contained in $\{m_C \leqslant h \leqslant M_C\}$. Similarly we show that $\mathcal{D}$ is contained in R_C, except that instead of considering minimal discs $h = constant$, we consider cylinders above geodesics of $\mathbb{H}^2$ and instead of vertical translation we use the fact that these cylinders foliate $\widetilde{PSL_2}(\mathbb{R})$. Note that the interior of $\mathcal{D}$ is strictly contained in the interior of $R_C \bigcap \{m_C \leqslant h \leqslant M_C\}$. □

The next proposition asserts the existence of a solution for the Dirichlet's problem for the minimal surface equation in $\widetilde{PSL_2}(\mathbb{R})$, over a convex bounded domain of $\mathbb{H}^2$ with prescribed continuous boundary data.

Proposition 5 **(Rado's Lemma in $\widetilde{\mathbf{PSL_2}}(\mathbb{R})$)** *If C admits a one-to-one projection onto a convex curve in $\mathbb{H}^2$, then the interior of $\mathcal{D}$ can be obtained as the image of a minimal section of π.*

Proof. Let C be a curve in $\mathbb{H}^2 \times \mathbb{R}$, as described above, which has a one-to-one projection onto a convex curve of $\mathbb{H}^2$. We want to prove that the interior of $\mathcal{D}$ is a graph over Ω, the open convex subset of $\mathbb{H}^2$ bounded by the $\pi(C)$.
Consider a vertical translate $\mathcal{D}'$ of $\mathcal{D}$, above $\mathcal{D}$, such that $\mathcal{D} \cap \mathcal{D}' = \emptyset$. We suppose that $\mathcal{D}^\circ$ is not a graph, so that there are two distinct points P and Q of $\mathcal{D}^\circ$, say P above Q, lying on the same fiber. Let P' and Q' be the corresponding translates of P and Q on $\mathcal{D}'$. We can translate $\mathcal{D}'$ down as to have $P \equiv Q'$. So at one point, when translating $\mathcal{D}'$ down, a translate $\mathcal{D}'$ will have a first point of contact with $\mathcal{D}$ without having $\mathcal{D} \equiv \mathcal{D}'$. By the maximum principle, this point of contact is not interior to both discs. So either the interior of one disc will touch the boundary of the other, or the boundaries of both discs touch at first. However, the above proposition shows that the interior of each disc lies in R_C°, and the boundaries lie on ∂R_C as they have convex projections to $\mathbb{H}^2$. So we are left with the only possibility that the first point of contact is a boundary point for both, which is a contradiction for the boundary is projected one-to-one into $S_\circ$ $\square$

In the next proposition we show that it is possible to claim existence of solutions when boundary data has a finite set of discontinuities. We will first prove the existence of a particular minimal graph which will be of use as a barrier later on.

Lemma 3 *Let $\mathcal{T}$ be an isosceles geodesic triangle in $\mathbb{H}^2$ with (open) sides $\mathcal{S}_i$ such that length$(\mathcal{S}_1)$ = length$(\mathcal{S}_2)$, and $c \in \mathbb{R}^*$. Let Δ denote the open bounded region of $\mathbb{H}^2$ bounded by $\mathcal{T}$. There exists a non-negative solution u of the minimal surface equation* (1) *defined in $\Delta \cup \{\mathcal{T}-$ vertices of $\mathcal{S}_i\}$ such that $u = 0$ on $\mathcal{S}_1$ and $\mathcal{S}_2$, and $u = c$ on $\mathcal{S}_3$.*

Proof. Consider such a triangle in $S_\circ$ and let $C \subset \widetilde{PSL_2}(\mathbb{R})$ be the Jordan curve formed by $\mathcal{S}_1$, $\mathcal{S}_2$, the translate of $\mathcal{S}_3$ to height $h = c$, and the two fiber segments joining the vertices of $\mathcal{S}_3$ to those of its translate. Let Σ be the interior of the solution of the Plateau problem for C. We shall show that Σ is a graph, thus showing the existence of our minimal section with the desired values on $\partial\mathcal{T}$.
Assume to the contrary that Σ is not a graph, so that there exist two points P and Q of Σ lying on the same fiber, say P above Q, with $d(P, Q) = d > 0$. Let f_ϵ be a family of isometries of $\mathbb{H}^2$ converging to the identity in C^1-topology, such that

$$f_\epsilon(\mathcal{S}_i) \bigcap \mathcal{C}(\mathcal{T}) = \emptyset,\ i = 1, 2.$$

Let $\tilde{f}_\epsilon$ denote the lift of f_ϵ to $\widetilde{PSL_2}(\mathbb{R})$ as explained in 2.4, and $\Sigma_{\epsilon,t} = \tilde{f}_\epsilon(\Sigma) + (0, 0, t),\ c.t > 0$. For $|t| \geq d$ and ϵ small enough, we have

$$\Sigma_{\epsilon,t} \cap C = \emptyset,\ \text{and}\ \partial\Sigma_{\epsilon,\mathrm{t}} \cap \Sigma = \emptyset.$$

We suppose, without loss of generality, that $c > 0$ and we remark that for ϵ small enough we'll have

$$\|\tilde{f}_\epsilon - Id_{\widetilde{PSL_2(\mathbb{R})}}\|_\infty < \frac{d}{2}.$$

To see the former equality we remark that the boundary of $\Sigma_{\epsilon,d}$ is composed of the arcs $C_{\epsilon,i} = \tilde{f}_\epsilon(\mathcal{S}_i) + (0,0,d)$, plus the fiber segments joining the extremities of $C_{\epsilon,1}$ to $C_{\epsilon,3}$ and $C_{\epsilon,2}$ to $C_{\epsilon,3}$. We can see that

$$h_{|C_{\epsilon,i}} > \tfrac{d}{2},\ (i = 1,\ 2),\ \text{and}\ h_{|C_{\epsilon,3}} > c + \tfrac{d}{2}.$$

Then these inequalities show that for ϵ small enough $\Sigma_{\epsilon,d}$ is above $z = \frac{d}{2}$ and hence

$$\Sigma_{\epsilon,d} \cap \mathcal{S}_i = \emptyset.$$

Moreover, $\Sigma_{\epsilon,d}$ lies in $\pi^{-1}(f_\epsilon(\mathcal{T}))$ by proposition 4, so that

$$\Sigma_{\epsilon,d} \cap \mathcal{C}_{\mathcal{S}_3} = \emptyset,$$

where $\mathcal{C}_{\mathcal{S}_3}$ is the cylinder above $\mathcal{S}_3$, completing the proof that $\Sigma_{\epsilon,d} \cap C = \emptyset$.
To show that $\partial\Sigma_{\epsilon,d} \cap \Sigma = \emptyset$, we first need to remark that $\Sigma \subset \pi^{-1}(\mathcal{T})$. This implies that Σ cannot intersect but possibly $C_{\epsilon,3}$ of $\partial\Sigma_{\epsilon,d}$. However the fact that $z_{|C_{\epsilon,3}} > c + \frac{d}{2}$ shows no intersection in this case either as Σ is below $z = c$.
Therefore,

$$\partial\Sigma_{\epsilon,d} \cap \Sigma = \emptyset.$$

Now the maximum principle implies that for ϵ small enough we have

$$\Sigma_{\epsilon,d} \cap \Sigma = \emptyset.$$

If we let $\epsilon \to 0$ we shall thus obtain that the limit surface, $\Sigma_d = \Sigma + (0,0,d)$, tangent to Σ at $P \in \Sigma$. By the maximum the two surfaces should be equal ; a contradiction. Therefore, Σ is a graph as was claimed. □

We now extend the result of proposition 5 to include Jordan curves containing finitely many vertical fiber segments.

Proposition 6 *Let Ω be a bounded convex domain in $\mathbb{H}^2$ and consider a finite set of boundary points of Ω. Let C denote the remaining boundary of Ω, which consists of a finite number of open arcs. Then there exists a solution of the minimal surface equation in Ω taking preassigned bounded continuous data on the arcs C.*

Proof. Let f be the bounded continuous data on C and f_n a bounded sequence of continuous functions on $\partial\Omega$ which converges uniformly to f on compacts of C. Let u_n be the solution of the minimal surface equation in Ω with boundary values f_n. Proposition 4 implies that the sequence u_n is uniformly bounded on compact sets of Ω, and hence by the compactness principle admits a subsequence which converges to a solution u in Ω.

The function u takes the values f on C as shown below using a standard barrier technique. Indeed, there exist barriers at each point of C, i.e., at each point P of C and for each pair of positive numbers K and δ, there exist a neighborhood V of P and a non-negative solution v in $V \cap \Omega$ such that

(i) $V \cap \Omega$ is contained in the geodesic disc of radius δ about P,
(ii) $v \geqslant K$ on $\partial V \cap \Omega$,
(iii) $v = 0$ at P.

We may take V to be an isosceles triangle, having its equal sides intersecting in Ω and tangent to $\partial\Omega$ at P on its third side, and v the solution in this triangle which takes values K on the equal sides and 0 on the third side. The existence of v is assured by lemma 3.

We shall show that u extends by continuity to f along C. Let $P \in \partial\Omega$, fix $\epsilon > 0$ and let v be a barrier at P defined in a triangle V as described above. As f_n is continuous at P then $\partial\Omega$ contains a neighborhood of P on which

$$f_n < f + \epsilon.$$

The continuity of f at P allows us to assume that in this neighborhood

$$f_n < f(P) + 2\epsilon$$

and hence in this neighborhood

$$f_n < v + f(P) + 2\epsilon.$$

We choose K such that

$$\sup_{\partial V \cap \Omega} (u_n) < K + f(P)$$

for the maximum principle would then imply the following inequality

$$u_n < v + f(P) + 2\epsilon \text{ in } V \cap \Omega$$

Taking $n \to \infty$ implies that

$$u(x) \leq v(x) + f(P) + 2\epsilon \text{ in } V \cap \Omega.$$

By a similar argument we obtain the inequality

$$u(x) \geq w(x) + f(P) - 2\epsilon \text{ in } V \cap \Omega,$$

where w is the barrier in the triangle V, chosen as for v above, except that w takes values $-K$ on the equal sides and 0 on the third side. The constant K is chosen such that

$$\inf_{\partial V \cap \Omega} (u_n) > -K + f(P).$$

Taking $\epsilon \to 0$ and $x \to P$, we get that $\lim_{x \to P} u(x) = f(P)$ and the proof is completed.□

2.6 The Conjugate Function

Let u be a solution of the minimal surface equation in a simply connected domain Ω. The equation

$$\mathrm{div}_{\mathbb{R}^2}\left(\frac{\lambda\alpha}{\mathrm{W}}\partial_{\mathrm{x}}+\frac{\lambda\beta}{\mathrm{W}}\partial_{\mathrm{y}}\right)=0$$

amounts to the fact that the differential

$$\omega=\frac{-\lambda\beta}{W}dx+\frac{\lambda\alpha}{W}dy$$

is exact in Ω. We may then consider the function ψ defined in Ω, such that $d\psi=\omega$, and we shall call it the conjugate function of u. The gradient of ψ, for the $\mathbb{H}^2$-metric, is

$$D\psi=\frac{-\beta}{\lambda W}\partial_x+\frac{\alpha}{\lambda W}\partial_y$$

and

$$|D\psi|_{\mathbb{H}^2}=\frac{\sqrt{\alpha^2+\beta^2}}{W}<1,$$

it follows that ψ is Lipschitz continuous and hence extends continuously to the closure of Ω and hence $d\psi$ may be integrated along boundary arcs of Ω regardless of the boundary values of u. The following is obvious

Lemma 4 *Let u be a solution of the minimal surface equation in a bounded domain $\Omega\subset\mathbb{H}^2$ and C a piecewise smooth curve lying in the closure of Ω. Then,*

$$\left|\int_C d\psi\right|\leqslant|C|,$$

where $|C|$ denotes the $\mathbb{H}^2$-length of C.

Moreover, if C is a simple closed curve then

$$\int_C d\psi=0.$$

We remark that if C lies in Ω, the fact that $|D\psi|<1$ implies that

$$\left|\int_C d\psi\right|<|C|.$$

We show next that this will be the case when C is a convex arc of the boundary of Ω, provided that u is continuous there.

Lemma 5 *Let u be a solution of the minimal surface equation in a domain Ω and C a convex arc of the boundary of Ω. If u is continuous on C then*

$$|\int_C d\psi| < |C|.$$

Proof. It is clearly enough to prove the result for a sub-arc of C; this allows us to assume, without loss of generality, that Ω is convex with u continuous on its boundary. Let C' denote the open sub-arc of the boundary which is complementary to C and let a be a real constant. The minimal surface equation admits a solution u^* which is equal to u on C' and $u + a$ on C, as guaranteed by the above results. We set

$$\tilde{u} = u^* - u \text{ and } \widetilde{\psi} = \psi^* - \psi.$$

Observe that $\tilde{u}_x = -\lambda(\alpha^* - \alpha)$ and $\tilde{u}_y = -\lambda(\beta^* - \beta)$, then integration by parts and a standard "approximation" at the end-points of C show that

$$\begin{aligned}
\int_{\partial\Omega} \tilde{u} d\widetilde{\psi} &= -\int_\Omega \Big[\tilde{u}_x\Big(\frac{\lambda\alpha}{W} - \frac{\lambda\alpha^*}{W^*}\Big) + \tilde{u}_y\Big(\frac{\lambda\beta}{W} - \frac{\lambda\beta^*}{W^*}\Big)\Big] dxdy \\
&= -\int_\Omega \lambda^2(\beta - \beta^*)\Big(\frac{\beta}{W} - \frac{\beta^*}{W^*}\Big) + \lambda^2(\alpha - \alpha^*)\Big(\frac{\alpha}{W} - \frac{\alpha^*}{W^*}\Big) dxdy \\
&= -\int_\Omega \Big\langle W\eta - W^*\eta^*, \eta - \eta^*\Big\rangle_{\widetilde{PSL_2(\mathbb{R})}} dA_{\mathbb{H}^2} \\
&= -\int_\Omega \frac{(W + W^*)}{2}(\eta - \eta^*)^2 dA_{\mathbb{H}^2},
\end{aligned}$$

where α^*, β^*, W^* and η^* are defined in terms the partial derivatives of u^* in the same fashion we defined α, β, W and η in terms of the partial derivatives of u. The field η^* is normal to the graph of u^*

The above computation then implies that

$$a\int_C d\tilde{\psi} < 0.$$

Using the fact that

$$|\int_C d\psi^*| \leqslant |C|$$

and giving a the values ± 1 complete the proof. ☐

Lemma 6 *Let Ω be a domain in $\mathbb{H}^2$ whose boundary contains a geodesic segment Γ. Suppose that $\partial\Omega$ is oriented so that the orientation on Γ coincides with that induced by the outward pointing normal to Γ. If u is a solution of the minimal surface equation in Ω assuming boundary value plus infinity on Γ then*

$$\int_\Gamma d\psi = |\Gamma|.$$

Proof. We consider the half plane model for the hyperbolic plane. We can suppose that $\Omega \subset \{x < 0, y > 0\}$ and that Γ is a segment of the geodesic $\{x = 0, y > 0\}$ of $\mathbb{H}^2$.
We remark that the $\mathbb{H}^2$-gradient of ψ, the conjugate function of u, is

$$D\psi = Rot_{\frac{\pi}{2}} d\pi(\eta),$$

where η is the upwards pointing unit normal to the graph Σ of u, and we show that η extends continuously to the boundary segment Γ.

We think of $\widetilde{PSL_2(\mathbb{R})}$ as a subset of $\mathbb{R}^3$ and we choose a sequence (p_n) of points with constant ordinates in Ω which converges to an interior point p of Γ. We set $\mu_n = d(p, p_n)$ and $q_n = (p_n, u(p_n))$ and we consider the affine transformations $h_n(X) = \frac{1}{\sqrt{\mu_n}}(X - q_n)$ on $\mathbb{R}^3$.
Let $\Sigma_n = h_n(\Sigma)$ and note that $0 \in \Sigma_n$, for all n, and that the normal η_n to Σ_n at the origin is the same as that of Σ at the point q_n. It is then enough to show $\eta_n(0)$ admits a limit as $n \to \infty$ and define $\eta(p)$ as this limit.

We admit for now that the sequence (A_n), A_n the second fundamental form of Σ_n for the euclidean metric, is uniformly bounded in a neighborhood of the origin, a claim we will prove below. Hence the sequence Σ_n converges on this neighborhood, up to a subsequence. As Σ_n is contained in $\{x \leqslant \sqrt{\mu_n}\}$ and asymptotic to the plane $\{x = \sqrt{\mu_n}\}$, the limit surface will be tangent to the plane $\{x = 0\}$ at the origin.

The sequence (N_n), N_n the normal to Σ_n for the euclidean metric at the origin, therefore converges to ∂_x. However the equality

$$\eta_n = \frac{G^{-1}N_n}{\sqrt{\langle G^{-1}N_n, N_n\rangle_{\mathbb{R}^3}}},$$

where the matrix G is such that $\langle X, Y\rangle_{\widetilde{PSL_2(\mathbb{R})}} = \langle GX, Y\rangle_{\mathbb{R}^3}$, implies that η_n is also convergent and this proves our claim that η extends by continuity to the interior of Γ.

The facts that at interior points of Γ

$$\begin{aligned}
\langle \eta, E_3\rangle_{\widetilde{PSL_2(\mathbb{R})}} &= \lim \langle \eta_n, E_3\rangle_{\widetilde{PSL_2(\mathbb{R})}} \\
&= \frac{\langle \partial_x, E_3\rangle_{\mathbb{R}^3}}{\sqrt{\langle G^{-1}\partial_x, \partial_x\rangle_{\mathbb{R}^3}}} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}\langle d\pi(\eta), e_2\rangle_{\mathbb{H}^2} &= \langle \eta, E_2\rangle_{\widetilde{PSL_2(\mathbb{R})}}\\ &= \frac{\langle \partial_x, E_2\rangle_{\mathbb{R}^3}}{\sqrt{\langle G^{-1}\partial_x, \partial_x\rangle_{\mathbb{R}^3}}}\\ &= 0\end{aligned}$$

imply that the extension of η to the boundary is such that

$$\langle d\pi(\eta), e_1\rangle_{\mathbb{H}^2} = -1,$$

e_1 being also the outwards pointing normal to Γ.

Now as $Rot_{\frac{\pi}{2}}$ preserves the metric on tangent spaces of $\mathbb{H}^2$ and as Γ is oriented by e_1 we obtain,

$$\begin{aligned}\int_\Gamma d\psi &= -\int_\Gamma \langle D\psi, e_2\rangle_{\mathbb{H}^2} ds\\ &= -\int_\Gamma \langle Rot_{\frac{\pi}{2}} d\pi(\eta), e_2\rangle_{\mathbb{H}^2} ds\\ &= -\int_\Gamma \langle d\pi(\eta), e_1\rangle_{\mathbb{H}^2} ds\\ &= |\Gamma|.\end{aligned}$$

To complete the proof we now estimate the second fundamental form A_n of Σ_n. Since $u \to \infty$ when $p \to \Gamma$ we may choose discs $\mathcal{D}(q_n, R)$ centered at q_n in Σ with intrinsic radius R independent of n, and since minimal graphs in $\widetilde{PSL_2(\mathbb{R})}$ are stable (see Remark 2.9), Schoen's curvature estimate implies that

$$|\widetilde{A}| \leqslant C \text{ in } \mathcal{D}(q_n, \tfrac{R}{2}),$$

where $\widetilde{A}$ is the second fundamental form of Σ for the $\widetilde{PSL_2(\mathbb{R})}$ metric and C is an absolute constant.

However, if N and A denote the normal and the second fundamental form of Σ with respect to the euclidean metric we have

$$\begin{aligned}\widetilde{A}(X,Y) &= \langle \overline{\nabla}_X Y, \eta\rangle_{\widetilde{PSL_2(\mathbb{R})}}\\ &= \frac{\langle \overline{\nabla}_X Y, N\rangle_{\mathbb{R}^3}}{\sqrt{\langle G^{-1}N, N\rangle_{\mathbb{R}^3}}}\\ &= \frac{1}{\sqrt{\langle G^{-1}N, N\rangle_{\mathbb{R}^3}}}\Big(\langle \nabla_X Y, N\rangle_{\mathbb{R}^3} + \langle \overline{\nabla}_X Y - \nabla_X Y, N\rangle_{\mathbb{R}^3}\Big),\end{aligned}$$

where ∇ is the Levi-Cevita connection of Σ for the Euclidean metric. Then $\widetilde{A}$ controls A as follows

$$A(X,Y) \leqslant \sqrt{\langle G^{-1}N, N\rangle_{\mathbb{R}^3}}\widetilde{A}(X,Y) - \langle \overline{\nabla}_X Y - \nabla_X Y, N\rangle_{\mathbb{R}^3}.$$

The tensor $\overline{\nabla}_X Y - \nabla_X Y$ can be easily seen to be controlled by $\|X\|$, $\|Y\|$ and the Christofel symbols of $\widetilde{PSL_2}(\mathbb{R})$ which shows that $|A|$ is bounded in a neighborhood of q_n. Then $\widetilde{A_n}$, the second fundamental form of Σ_n with respect to $\widetilde{PSL_2}(\mathbb{R})$ metric, is bounded by $C\sqrt{\mu_n}$ in the disc $\mathcal{D}(0, \frac{R}{2\sqrt{\mu_n}})$. In a similar fashion, one obtains the following estimates

$$A_n(X,Y) \leqslant \sqrt{\langle G^{-1}N_n, N_n\rangle}\widetilde{A}_n(X,Y) - \langle \overline{\nabla}_X Y - \nabla_X Y, N_n\rangle.$$

which imply that $(A_n)_n$ is uniformly bounded in a neighborhood of the origin and the proof is completed. $\square$

Lemma 7 *Let Ω be a domain in $\mathbb{H}^2$ as in Lemma* 6 *and let (u_n) be a sequence of solutions of* (1) *in Ω. Assume that each (u_n) is continuous in $\Omega \cup \Gamma$ and that (u_n) diverges uniformly to infinity on compact subsets of Γ while remaining uniformly bounded on compact subsets of Ω. Then*

$$\lim_{n\to\infty} \int_\Gamma d\psi_n = |\Gamma|.$$

On the other hand, if the sequence diverges uniformly to infinity on compact subsets of Ω while remaining uniformly bounded on compact subsets of Γ, then

$$\lim_{n\to\infty} \int_\Gamma d\psi_n = -|\Gamma|.$$

Proof. We follow the same lines of proof as in Lemma 6 except that we choose the points $q_n = (p_n, u_n(p_n))$ instead, where (p_n) is in Ω and converges to an interior point of Γ. We consider the surfaces $\mathcal{S}_n = h_n(\Sigma_n)$ in $\mathbb{R}^3$, where Σ_n is the graph of u_n and h_n is as defined in the proof of Lemma 6, for purposes similar to those in that proof. We let A_n denote the second fundamental form of $\mathcal{S}_n$ and η_n the normal to $\mathcal{S}_n$ at the origin, which is the same as that to Σ_n at q_n.

The facts that u_n is continuous in $\Omega \cup \Gamma$ and that the sequence (u_n) diverges uniformly on compacts of Γ, allow us to choose discs centered at q_n on Σ_n, of radius R independent of n, as in the proof of Lemma 6.

Moreover, we note that the sequence (u_n) converges to a solution in Ω with u taking the value $+\infty$ on Γ. This fact together with Schoen's curvature estimate for each Σ_n, imply in a similar way as in the proof of Lemma 6 that $(A_n)_n$ is uniformly bounded in a $D(0,R)$. The sequence (η_n) can then be proved to converge to a horizontal vector η along Γ with

$$\langle d\pi(\eta), e_1 \rangle = -1$$

and then

$$\lim_{n\to\infty} \int_\Gamma d\psi_n = -\int_\Gamma \langle d\pi(\eta), e_1 \rangle ds = |\Gamma|$$

To prove the second part of the lemma we make the obvious adjustments to the proof and further details are left to the reader. □

2.7 The Monotone Convergence Theorem

Later existence results depend on the limit behavior of monotone sequences of solutions of the minimal surface equation. In this section we develop the necessary tools to deal with these sequences. These are similar, as well as the last two sections above, to the results in [16].

Lemma 8 (**Straight Line Lemma**) *Let $\Omega \subset \mathbb{H}^2$ be a bounded domain whose boundary consists of a geodesic segment γ and an arc C, with Ω lying on one side of γ. Then for any compact $K \subset \Omega$ there exists N, depending only on the distance from K to γ, such that*

$$m - N \leqslant u \leqslant M + N \text{ in } K,$$

for any solution u of the minimal surface equation (2.1) *which is bounded in $\overline{\Omega}$, with $m \leqslant u \leqslant M$ on C.*

Proof. Let f_1 and f_2 be two isometries of $\mathbb{H}^2$ sending the positive y-axis to the geodesic Γ which contains γ such that the image of the quadrant $Q_1 = \{(x,y)|x > 0, y > 0\}$ by f_1 will contain Ω, and the image of the quadrant $Q_2 = \{(x,y)|x < 0, y > 0\}$ will contain Ω. Let $O = f_1(Q_1) = f_2(Q_2)$ and note that the minimal graphs of example 2 can be used to obtain a positive solution and a negative solution of the minimal surface equation in O, which take respectively the value $+\infty$ and $-\infty$ on Γ. Simply, let $\tilde{f}_1$ and $\tilde{f}_2$ denote the respective lifts of f_1 and f_2 to $PSL_2(\mathbb{R})$ and consider the images by $\tilde{f}_1$ and $\tilde{f}_2$ of the graphs in example 2, which correspond to $C = 1$ and defined over Q_1 and Q_2 respectively. We obtain two graphs on O which, up to vertical translations, have the desired properties. Assume these graphs to be those of solutions $v_1 \geq 0$ and $v_2 \leq 0$ of (2.1).
Let u a solution of the minimal surface equation in Ω with $m \leqslant u \leqslant M$ on C. Then on the boundary of Ω we shall have

$$m + v_2 \leqslant u \leqslant M + v_1.$$

The maximum principle then implies that the inequalities hold in Ω. Now, for any compact K of $\Omega \cup C$, let $N = \max\{\max_K v_1, \left|\min_K v_2\right|\}$ which depends only on the distance from K to γ. We clearly have that

$$m - N \leqslant u \leqslant M + N \text{ in } K,$$

which concludes our proof. □

Remark 13 *One direct consequence of this lemma is that no solution of the minimal surface equation can take infinite values on a non-geodesic boundary arc of a convex domain. Assume to the contrary that there exists a solution u of (2.1) in a convex domain Ω taking the value $+\infty$ ($-\infty$) on a non-geodesic open boundary arc C. By restricting ourselves to proper parts of C we may assume U, the convex hull of C in Ω, bounded by C and its end points and an open geodesic segment γ contained in Ω. We shall obtain a contradiction by showing that u must be equal to $+\infty$ ($-\infty$) in U. Let $a = \inf_\gamma u$ ($= \sup_\gamma u$) which may be assumed a positive (negative) real number (if we restrict ourselves to proper parts of C). For each n, let u_n be the solution of the minimal surface equation in U taking the values n ($-n$) on C and a on γ. By the maximum principle, we have then $u_n \leqslant u$ ($u_n \geqslant u$) in U. Hence by the Straight Line Lemma we have that on each compact in $U \cup C$ and for each n, $n - N \leqslant u_n \leqslant u$ ($-n + N \geqslant u_n \leqslant u$) with N independent of n. Letting $n \to \infty$ implies that u has infinite values in U which is absurd.*

The following two theorems are essential for studying convergence of monotone sequences of solutions.

Theorem 8 (**Monotone Convergence Theorem**) *Let (u_n) be a monotonically increasing sequence of solutions of the minimal surface equation in a domain Ω. If the sequence is bounded at a point $p \in \Omega$, then there exists a non-empty open set $U \subset \Omega$ such that the sequence (u_n) converges to a solution in U, and diverges to infinity on the complement of U. The convergence is uniform on compacts of U, and the divergence is uniform on compacts of $V = \Omega - U$.*

Proof. Assume that $|u_0| \leq c$ near p, and consider the sequence of non-negative solutions (v_n) such that $v_n = u_n + c$. Hence, each $W_n(p) \leq C_n$, where C_n is the constant given by theorem 6. Then theorem 7 implies that each W_n is bounded in a disc centered at p and of radius R_n, with R_n depending on $u_n(p)$, $d(p, \partial\Omega)$ and bounds of λ and its derivatives. As $(u_n(p))$ is bounded, then we can find a disc D centered at p on which (W_n) is uniformly bounded. The mean value theorem then implies that (u_n) is then uniformly bounded in this disc. The compactness principle therefore implies that (u_n) has a convergent subsequence and as (u_n) is monotone it converges on this disc. The compactness principle implies also that the limit is a solution of the minimal surface equation and so U is a non-empty open set. The divergence is uniform on compacts of V as the sequence is monotonically increasing. The divergence set V is by no means arbitrary, it has a very particular geometric structure. We resume the properties of V in the following

Theorem 9 (**Divergence Set Structure Theorem.**)*Let (u_n) be a monotonically increasing sequence of solutions in Ω. If the divergence set $V \neq \emptyset$, then int$(V) \neq \emptyset$,*

and ∂V is composed of non-intersecting geodesic segments of Ω and possibly parts of $\partial\Omega$. Moreover, no two interior geodesic segments of ∂V can have a common end point at a convex corner of V, nor any component of V consist only of a geodesic segment of Ω.
Furthermore, if Ω is bounded in part by a convex arc C with each u_n continuous in $\Omega \cup C$ and (u_n) either diverges to infinity on C or remains uniformly bounded on compacts of C, then no interior geodesic segment Γ forming part of the boundary of V can terminate at an interior point of C.

For the proof of this theorem, one can employ the lemmas of section 6 above in ways similar to those in the proofs of Lemma 5 and Lemma 6 in [16]. In fact, Remark 13 above implies that, if $V \neq \emptyset$, ∂V consists of non intersecting geodesic segments of Ω and possibly parts of the boundary of Ω. To prove that no component of ∂V is only a geodesic segment T of Ω, one applies Lemma 6 above to Ω_1 and Ω_2, the components of Ω on either side of T. A contradiction is obtained since in Ω_1, say, one obtains

$$\lim_{n\to\infty} \int_T d\psi_n = |T|$$

and in Ω_2 one obtains

$$\lim_{n\to\infty} \int_T d\psi_n = -|T|\,.$$

To see that no interior geodesic segments of ∂V can have a common end point, we notice that Remark 13 above implies that such a point must be in $\partial\Omega$. We suppose then ∂V admits two geodesic segments T_1 and T_2 in Ω with a common end point Q in $\partial\Omega$ and we choose two points Q_1 and Q_2 on T_1 and T_2 respectively, so that the open geodesic triangle Δ, with vertices Q, Q_1 and Q_2, lie in Ω. By Lemma 4 above,

$$\int_{QQ_1} d\psi_n + \int_{Q_1Q_2} d\psi_n + \int_{Q_2Q} d\psi_n = 0.$$

The triangle may lie in U or V, since no component of ∂V is only a geodesic segment. In the former case, one applies Lemma 7 above to obtain

$$\lim_{n\to\infty} \int_{QQ_1} d\psi_n = |QQ_1| \text{ and } \lim_{n\to\infty} \int_{QQ_2} d\psi_n = |QQ_2|$$

assuming that QQ_1Q_2 determines the positive orientation of Δ. However, Lemma 4 implies

$$\left| \int_{Q_1Q_2} d\psi_n \right| \leq |Q_1Q_2|$$

which is a contradiction with the triangle inequality in $\mathbb{H}^2$. If Δ lies in V one obtains a similar contradiction by applying the second part of Lemma 7. To prove the second part of Theorem 6, we notice that if C is not geodesic, Lemma 8 implies that on compacts in the convex hull of C

$$\min_C(u_n) - N \leq u_n \leq \max_C(u_n) + N$$

with N independent of n, and the proof of the claim is immediate since the above inequality implies that the interior of the convex hull of C lies either in U or in V. We then assume that C is geodesic, and that Γ terminates at an interior point Q of C. Suppose first that the sequence diverges on C. Let P be a point of Γ, and choose a point R on C such that the geodesic segment RP lies in U. The results we have proved in the first part of the theorem allow this choice. We apply Lemma 4 and Lemma 7 in the triangle QPR in a fashion similar to that in the triangle Δ, to obtain a contradiction with the triangle inequality in $\mathbb{H}^2$. In case the sequence remains uniformly bounded on compacts of C, a similar contradiction results by choosing the segment RP in V. $\square$

2.8 A Jenkins-Serrin Type Theorem

This is theorem 4 stated in the introduction. We note that a section s of $\pi : \widetilde{PSL_2}(\mathbb{R}) \to \mathbb{H}^2$ takes the value $+\infty$ ($-\infty$ resp.) on a geodesic segment A_i (B_j resp.) if the image by s of each geodesic $t \to \gamma(t)$ of Ω ending at A_i (B_j resp.) gets out of every compact and if $\langle \gamma'(t), \xi \rangle > 0 (< 0, resp.)$, where $\xi = \partial_z$ in our model of $\widetilde{PSL_2}(\mathbb{R})$.

As was remarked above, having fixed the model for $\widetilde{PSL_2}(\mathbb{R})$ the existence of the section s on Ω with the prescribed boundary data is equivalent to the existence of a real function u defined in Ω with corresponding data on the boundary. The function u is constructed as a limit of monotone sequence of solutions of the minimal surface equation whose behavior is studied using the monotone convergence theorem, the divergence set structure theorem and the properties of the differential $d\psi$ corresponding to u. Once the convergence is established, we need to show that the limit will assume the appropriate boundary values. This will be assured by the Boundary Values Lemma below.

We proceed to prove the existence of particular solutions of (2.1) which will be used as barriers in the proof of the Boundary Values Lemma.

Lemma 9 *Let $\mathcal{P}$ be a convex quadrilateral in $\mathbb{H}^2$, formed by geodesic segments A_1, A_2, C_1 and C_2 such that $A_1 \cap A_2 = \emptyset$ and $|A_1| + |A_2| < |C_1| + |C_2|$. Then there exists a solution of* (2.1) *in $\mathcal{P}$ which takes the boundary values $+\infty$ on $A_1 \cup A_2$ and non-negative values on $C_1 \cup C_2$.*

Proof. Let $\mathcal{P}$ be a convex quadrilateral in $\mathbb{H}^2$, formed by geodesic segments A_1, A_2, C_1 and C_2 such that $A_1 \cap A_2 = \emptyset$ and $|A_1|+|A_2| < |C_1|+|C_2|$. Let u_n be the solution of the minimal surface equation in $\mathcal{P}$ taking boundary values n on each A_i and 0 on each C_i. The sequence u_n is seen to converge to a solution u in $\mathcal{P}$ as follows. Let $\mathcal{V}$ denote the divergence set and remark that either the interior of $\mathcal{V}$ is equal to that of $\mathcal{P}$, or otherwise by Theorem 9 an interior geodesic segment bounding $\mathcal{V}$ must have its endpoints from amongst those of the A_i's.

The interior of $\mathcal{V}$ cannot be equal to that of $\mathcal{P}$ for otherwise :

$$\int_{A_1 \cup\ A_2} d\psi_n + \int_{C_1 \cup\ C_2} d\psi_n = 0$$

and then one takes the limit as $n \to +\infty$ and uses Lemma 7 and Lemma 4 to obtain

$$\int_{C_1 \cup\ C_2} d\psi_n = -\big(|C_1| + |C_2|\big) \text{ and } \int_{A_1 \cup\ A_2} d\psi_n \leq |A_1| + |A_2|.$$

This implies that $|A_1| + |A_2| \geq |C_1| + |C_2|$ which is not true.

Thus assume that $\mathcal{V}$ is non-empty and bounded by a geodesic triangle Δ whose vertices are endpoints of the A_i's. Let δ denote the perimeter of Δ. One would obtain

$$\int_{\Delta - A_i} d\psi_n + \int_{A_i} d\psi_n = 0.$$

Again passing to the limit and using Lemma 7 and Lemma 4 the following holds

$$\int_{\Delta - A_i} d\psi_n = -(\delta - |A_i|) \text{ and } \int_{A_i} d\psi_n \leq |A_i|,$$

which leads to a contradiction with the triangle inequality.

Therefore, $\mathcal{V} = \emptyset$ and the sequence (u_n) converges on compact sets of $\mathcal{P}$ to a solution of (2.1). We note that since (u_n) is increasing (by the maximum principle), u takes the value $+\infty$ on the segments A_i. Although at this point we do not know yet that $u = 0$ on the C_i's, a fact which we will be able to prove later, we remark that $u \geq 0$ on each C_i. $\square$

Lemma 10 *Let $\mathcal{P}$ be a convex quadrilateral in $\mathbb{H}^2$ formed by geodesic segments A_1, A_2, C_1 and C_2 such that $A_1 \cap A_2 = \emptyset$. If $|A_1| + |A_2| < |C_1| + |C_2|$, then there exists a solution v of the minimal surface equation in $\mathcal{P}$ taking the boundary values $+\infty$ on $A_1 \cup\ A_2$ and has bounded values on $C_1 \cup C_2$.*

Proof. Let $\widetilde{C_i}$ be a horizontal lift of C_i to $\widetilde{PSL_2(\mathbb{R})}$ and let u_n be the solution of the minimal surface equation in $\mathcal{P}$ taking boundary values n on each A_i and boundary values given by $\widetilde{C_i}$ on C_i. One may translate vertically each of the $\widetilde{C_i}$, so that each

$u_n \geq 0$. The sequence u_n is increasing and converges to a solution u in $\mathcal{P}$ by arguments similar to those in Lemma 9.
The limit u takes the boundary value $+\infty$ on each A_i as the sequence (u_n) is increasing. The boundary values of u on C_i are given by $\widetilde{C}_i$ and this follows by standard barrier techniques, as in the proof of Proposition 6, once we show the sequence (u_n) to be uniformly bounded near each point of C_i. We complete the proof by showing this last point.
As $\widetilde{C}_i$ is a horizontal geodesic, observations in [1] ensures that the graph of u_n extends by symmetry about $\widetilde{C}_i$ to a graph (a graph since otherwise by the maximum principle, the surface obtained by symmetry would coincide with the cylinder $\pi^{-1}(C_i)$).
Let p be a point of C_i and choose a sufficiently small geodesic rectangle R as in Lemma 9, which has two of its sides orthogonal to C_i and which contains p in its interior. Let v denote the solution of (2.1) in R taking the values $+\infty$ on the sides orthogonal to C_i and non-negative values on the other two sides, say S_j, $1 \leq j \leq 2$. The existence of v is assured by Lemma 9. The maximum principle then implies that for each n, $u_n \leq v + M$ over R, where $M = \sup |u_n|$ and the supremum is taken over the S_j's. One considers a small neighborhood of p in R, and the preceding inequality proves that u_n is bounded around p. □

Lemma 11 (**Boundary Values Lemma**) *Let Ω be a domain and C a compact convex arc in its boundary. Let (u_n) be a sequence of solutions of the minimal surface equation, which converges uniformly on compacts of Ω to a solution u. Suppose that, on the one hand, each (u_n) is continuous in $\Omega \cup C$ and that the boundary values converge uniformly on compacts of C to a limit function f. Then u is continuous in $\Omega \cup C$ and takes the values f on C. If C, on the other hand, were a geodesic segment where the boundary values diverge uniformly to infinity, then u will take on the boundary value infinity on C.*

Proof. For the first part where the boundary values of (u_n) converge uniformly on compact subsets of C it suffices to show the sequence (u_n) uniformly bounded in the neighborhood of any interior point of C and then employ a standard argument of the theory of barriers (again similar to that in the proof of proposition 6 above). If C is not geodesic then the result follows by the Straight Line Lemma. In case C is a geodesic segment, the preceding lemma furnishes the ingredient necessary to show the required boundedness of (u_n) near interior points of C in a fashion similar to that of Lemma 7 in [16] or the corresponding Boundary Values Lemma in [5].
The part where C is geodesic and (u_n) taking infinite values there can be proved in a fashion similar to that of Lemma 8 in [16]. However, to prove (u_n) bounded from below as is done in [16] we may prove a lemma similar to Lemma 10 above except that the solution takes values $-\infty$ on the sides A_i. Then we follow the same lines of proof of the Boundary Values Lemma in [5]. □

Remark 14 *Let $\mathcal{P}$ be a geodesic rectangle as in Lemma* 10. *In order to prove the existence of a solution of* (2.1) *in $\mathcal{P}$ taking bounded values on the C_i's and values*

$-\infty$ on the A_i's, one can proceed as follows. Let r denote the reflection of $\mathbb{H}^2$ in A_1, and $\tilde{r}$ its lift to $\widetilde{PSL_2}(\mathbb{R})$. Consider the image $\mathcal{P}'$ of $\mathcal{P}$ by r, and find by Lemma 10 a solution v in $\mathcal{P}'$ taking the values $+\infty$ on A_1 and $r(A_2)$, and bounded values on each $r(C_i)$. The image by $\tilde{r}$ of the graph of v is the graph of a solution u of (2.1) over $\mathcal{P}$, which takes the sought boundary values.

Having developed the necessary machinery in this paper, the existence part of Theorem 4 could be proved following the same lines of proof in [16] and [25]. To see that the conditions in Theorem 4 are necessary, we let u be a solution of the minimal surface equation in a domain Ω and we consider a polygon $\mathcal{P}$, such that Ω and $\mathcal{P}$ are as described in that theorem. By Lemma 4 above

$$\int_{\mathcal{P}-\{A_i\in\mathcal{P}\}} d\psi + \sum_{A_i\in\mathcal{P}} \int_{A_i} d\psi = 0,$$

with $\mathcal{P}$ oriented by the outward pointing normal. Lemma 6 implies that

$$\sum_{A_i\in\mathcal{P}} \int_{A_i} d\psi = \alpha,$$

and if $\mathcal{P} \neq \partial\Omega$ then Lemma 5 implies that

$$\int_{\mathcal{P}-\{A_i\in\mathcal{P}\}} d\psi < \gamma - \alpha.$$

If $\mathcal{P} = \partial\Omega$, which is possible only if the family $\{C_i\} = \emptyset$, then again by Lemma 6, one would obtain

$$\int_{\mathcal{P}-\{A_i\in\mathcal{P}\}} d\psi = -\beta.$$

This argument shows that the conditions $2\alpha < \gamma$ for all possible polygons $\mathcal{P} \neq \partial\Omega$ chosen as in Theorem 4, and that $\alpha = \beta$ when $\mathcal{P} = \partial\Omega$ are necessary. A similar argument shows that the conditions on the segments B_i are necessary as well.

To show that the conditions of Theorem 4 are sufficient, we employ the Monotone Convergence Theorem, the Divergence Structure Theorem and the lemmas of sections 6 through 8 in the same fashion as in [16] or [25]. We furnish only a sketch of the proof and we refer the reader to section 5 in [16] for further details, where the constructions of solutions held in that paper carry word for word to our case. The proof can be broken down into proving existence of solutions of Dirichlet problems related to the one stated in Theorem 4.

Step 1. We consider the Dirichlet problem in Theorem 4, and we suppose that the family $\{B_i\}$ is empty. Assume also that the assigned data on the arcs $\{C_i\}$ is bounded below. Then the conditions $2\alpha < \gamma$ for each simple closed polygon $\mathcal{P}$ whose vertices

are chosen from among the endpoints of the A_i's are sufficient for the existence of a solution.
Step 2. We consider the Dirichlet problem in Theorem 4, and we suppose that the family $\{C_i\} \neq \emptyset$. Then the conditions $2\alpha < \gamma$ and $2\beta < \gamma$ for each simple closed polygon $\mathcal{P}$ whose vertices are chosen from among the endpoints of the A_i's and the B_i's are sufficient for the existence of a solution.
Step 3. We consider the Dirichlet problem in Theorem 4, and we suppose the family $\{C_i\} = \emptyset$. Then the conditions $\alpha = \beta$ when $\mathcal{P} = \partial\Omega$ and $2\alpha < \gamma$ and $2\beta < \gamma$ for each simple closed polygon $\mathcal{P}$ whose vertices are chosen from among the endpoints of the A_i's and the B_i's are sufficient for the existence of a solution.

To complete the proof of Theorem 4, we next give a proof of the uniqueness, which is up to an additive constant when the family $\{C_i\} = \emptyset$, inspired by [4].

Proof of uniqueness. Let u_1 and u_2 be two different solutions of the minimal surface equation with the same boundary data (possibly infinite). If $\{C_i\} = \emptyset$ we suppose that $u_1 - u_2$ is not a constant. Note that either of the subset of Ω, $\{u_1 > u_2\}$ or $\{u_1 < u_2\}$ is non-empty . We suppose without loss of generality that $\{u_1 > u_2\} \neq \emptyset$ and we choose ϵ small enough so that $\Omega_\epsilon = \{u_1 - u_2 > \epsilon\}$ is non-empty and that $\partial\Omega_\epsilon$ is regular.
We consider the closed differential $d\Psi = d\psi_1 - d\psi_2$, ψ_1 and ψ_2 the conjugate functions of u_1 and u_2 respectively, and we obtain a contradiction by showing that $\int_{\partial\Omega_\epsilon} d\Psi \neq 0$.
As u_1 and u_2 have the same boundary data $\partial\Omega$ does not intersect $\cup C_i$, besides Lemma 6 implies that $d\Psi = 0$ on $\cup A_i \bigcup \cup B_j$. Then the only part of $\partial\Omega_\epsilon$ which contributes to the integral $\int_{\partial\Omega_\epsilon} d\Psi$, denoted $\widetilde{\partial\Omega_\epsilon}$, is that contained in Ω_ϵ defined by $u_1 - u_2 = \epsilon$. Then the vector

$$v = Rot_{\frac{\pi}{2}}\Big(\nabla(u_1 - u_2)\Big) = -\lambda(\beta_1 - \beta_2)\partial_x + \lambda(\alpha_1 - \alpha_2)\partial_y$$

is tangent to $\widetilde{\partial\Omega_\epsilon}$ and the integral $\int_{\partial\Omega_\epsilon} d\Psi$ reduces to integrating $d\Psi.v$. However, a computation similar to that of lemma 5 shows that

$$d\Psi.v = \lambda^2 \frac{(W_1 + W_2)}{2}(\eta_1 - \eta_2)^2$$

which is a positive quantity (η_i is the normal to the graph of u_i). This leads to a contradiction and the proof is completed. $\square$

Troisième partie

Surfaces Minimales dans l'Espace Euclidien

Chapitre 3

Construction of Triply Periodic Minimal Surfaces

3.1 Introduction

During the middle of the nineteenth century, H.A. Schwarz carried an intensive investigation of periodic minimal surfaces and was able to construct five triply periodic ones. A minimal surface in the euclidean space is said to be triply periodic if it is invariant under three independent translations. His method consisted of spanning a disc-type minimal surface into a non-planar polygonal boundary, and then reflecting this surface across its boundary lines.
In the 1970's, the physicist and crystallographer Alan Schoen, discovered many triply periodic minimal surfaces and constructed models of them. However, his study of these surfaces was a bit sketchy and thus, among mathematicians, there remained doubts whether all details could be filled in. It did not take long until Hermann Karcher established rigorously the existence of all of Schoen's surfaces, and constructed whole families of newly found triply periodic embedded minimal surfaces, by applying his so called "Conjugate Plateau Constructions" [18].

In this paper, we will construct families of triply periodic embedded minimal surfaces by gluing simply periodic ones with Scherk type ends. A Scherk type end is one which is asymptotic to a vertical half-plane. Our construction is motivated by the following observation : it is known that the Schwarz P-surface can be deformed in a one parameter family P_ε, with periods $(1,1,0)$, $(1,-1,0)$ and $(0,0,\varepsilon)$, such that when $\varepsilon \to 0$, P_ε converges to the set $\mathcal{T} \times \mathbb{R}$, where $\mathcal{T}$ is the standard tiling of the plane by unit squares. We reverse the process, given a tiling $\mathcal{T}$ of the plane, which is invariant under two independent translations, we construct a family M_ε of triply periodic embedded minimal surfaces, each of which has a horizontal period equal to that of the tiling, which converges to $\mathcal{T} \times \mathbb{R}$. Of course, we do not expect the construction to work for arbitrary periodic tilings as will be seen later.

Roughly speaking, the idea underlying the gluing process is the following : from a distance, a Karcher saddle tower (see Section 3.3 below) is seen as a set of vertical half-planes intersecting at a common line. We assume that all of the saddle towers to be considered, admit the same vertical period $(0, 0, T)$, which without loss of generality we normalize to $(0, 0, 2\pi)$. We scale each of the saddle towers by a factor of ε^2 (!), so that the vertical period is $P(\varepsilon) = (0, 0, \varepsilon^2 T)$. Then, given a tiling $\mathcal{T}$ of $\mathbb{R}^2$, invariant under two independent translations, we place a scaled Karcher saddle tower at each of its vertices in such a way that, the number of wings of the saddle tower is equal to the number of edges ending at the vertex where it is placed, and each wing goes along an edge. For each edge we glue the corresponding wings of the scaled saddle towers placed at its ends, resulting in a triply periodic surface whose horizontal period is that given by the tiling and a vertical period $P(\varepsilon)$. It is natural that we require the surfaces to be symmetric with respect to the horizontal plane as it is the case for the saddle towers under consideration. We let L_ε denote the lattice generated by the periods of the tiling and P_ε.

The construction will be accomplished by furnishing Weierstrass data on appropriate Riemann surfaces, where we employ Weierstrass representation of a minimal surface in its simplest form. On a Riemann surface Σ_ε, we furnish three holomorphic differentials, say ϕ_1^ε, ϕ_2^ε, ϕ_3^ε, verifying the conditions

$$\phi_1^{\varepsilon\,2} + \phi_2^{\varepsilon\,2} + \phi_3^{\varepsilon\,2} = 0 \text{ } on \text{ } \Sigma_\varepsilon \tag{3.1}$$

$$|\phi_1^\varepsilon|^2 + |\phi_2^\varepsilon|^2 + |\phi_3^\varepsilon|^2 > 0 \text{ } on \text{ } \Sigma_\varepsilon, \tag{3.2}$$

and we set $X_\varepsilon(p) = Re \int^p \Phi_\varepsilon$, where $\Phi_\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon, \phi_3^\varepsilon)$.

The inequality (2) implies that X_ε is an immersion. The equation (1) amounts to the conformality of the harmonic map X which, together with (3.2), implies that X_ε is a minimal immersion. Up to translations, X_ε immerses Σ_ε into a triply periodic minimal surface whose period is L_ε provided that we have

$$Period_\alpha(\Phi_\varepsilon) = Re \int_\alpha \Phi_\varepsilon \in L_\varepsilon \tag{3.3}$$

for all closed cycles α on Σ_ε.

Inspired by the work of Martin Traizet [34] and [35], we perform the gluing by opening the nodes of singular Riemann surfaces with nodes and we invest the regeneration of the regular differential forms they carry into holomorphic forms. We adjust some parameters underlying the construction, by applying the implicit function theorem, so that (3.1), (3.2) and (3.3) hold.

The paper is organized as follows. In section 2, we fix notation for the tilings under consideration and define what a balanced and a rigid tiling is, and we furnish sufficient conditions for rigidity. In section 3 we remind the reader of the needed properties of Karcher saddle towers then we state our main result and give some examples. In section 4, we present the machinery we will employ throughout our construction and we write equations corresponding to (1), (2) and (3) in terms of the parameters underlying the construction introduced in that same section. In section 5, we apply the implicit function theorem to show that we may choose our parameters as to solve the equations presented in the preceding section. Finally, we complete the proof of the main result of the paper and we prove that the surfaces obtained are embedded.

3.2 Notions On Tilings

3.2.1 Definitions and Notations

We consider a periodic tiling $\mathcal{T}$ of the plane by straight edge polygons, which is invariant by two independent translations T_1 and T_2. The bounded domains enclosed by these polygons are said to be the faces of $\mathcal{T}$, and whose edges and vertices are said to be the edges and vertices of $\mathcal{T}$ respectively. Two faces of $\mathcal{T}$ are said to be adjacent if they have a common edge. We do not suppose T_1 and T_2 to be the smallest periods of $\mathcal{T}$ and we let Γ denote the group generated by these two vectors.

The tiling $\mathcal{T}$ projects onto a tiling of the quotient $\mathbb{R}^2/\Gamma$ which we also denote by $\mathcal{T}$ and which we suppose to have finitely many vertices. We shall assume that each vertex is incident with at least three edges and that each edge bounds exactly two faces. We say that e ends at v if v is an extremity of e, and we write $e = vv'$ for an edge e of $\mathcal{T}$ whose extremities are v and v'.

In the quotient, let $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{F}$ denote respectively, the set of vertices, the set of edges and the set of faces of $\mathcal{T}$. We consider the sets

$$\mathcal{E}_v = \{e : e \textit{ ends at } v\} \text{ and } \mathcal{E}_f = \{e : e \in \partial f\}$$

respectively the set of edges ending at a given vertex $v \in \mathcal{V}$ (*i.e.* having v as an extremity) and the set of edges bounding a given face $f \in \mathcal{F}$. We also consider

$$\mathcal{V}_f = \{v : v \text{ is a vertex of } f\},$$

the set of vertices of a face $f \in \mathcal{F}$. We let n_v, n_e and n_f denote respectively the cardinal of $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{F}$. Since the quotient $\mathbb{R}^2/\Gamma$ has an Euler characteristic equal to 0, we obtain the following equation

$$n_v - n_e + n_f = 0.$$

For $v \in \mathcal{V}$ we denote by $d(v)$ the cardinal of $\mathcal{E}_v$, which is the number of edges ending at v, and we say $d(v)$ is the degree of v.

For later purposes, it is convenient to prescribe a sign to each vertex and to each edge of $\mathcal{T}$. We denote the arbitrarily prescribed sign of a vertex v by $\sigma(v) = \pm 1$ and for $e = vv'$ we let $\sigma(e) := \sigma(v)\sigma(v')$.

Proposition 7 *Each face of $\mathcal{T}$ has an even number of edges with a negative sign.*

Proof. Let $f \in \mathcal{F}$,

$$\begin{aligned}
\prod_{e \in \partial f} \sigma(e) &= \prod_{\{e=vv' : e \in \partial f\}} \sigma(v)\sigma(v') \\
&= \prod_{v \in \mathcal{V}_f} \sigma(v) \prod_{v' \in \mathcal{V}_f} \sigma(v') \\
&= \Big(\prod_{v \in \mathcal{V}_f} \sigma(v) \Big)^2 \\
&= 1.
\end{aligned}$$

which ends the proof. $\square$

We identify $\mathbb{R}^2$ and the complex plane $\mathbb{C}$ and we index the components of the vectors in $\mathbb{C}^{n_v}$ by $v \in \mathcal{V}$. We follow a similar convention for $\mathbb{C}^{n_e}$ and $\mathbb{C}^{n_f}$. For each vertex $v \in \mathcal{V}$, we consider the lifts $e \in \mathcal{E}_v$ to edges of the tiling in the plane ending at some lift z_v of v. Then the resultant

$$\mathcal{R}_v = \sum_{e \in \mathcal{E}_v} \frac{z_{v'} - z_v}{|z_{v'} - z_v|}$$

depends only on v.

We deform the quotient tiling $\mathcal{T}$ by $\mathbf{h} = (h_v) \in \mathbb{C}^{n_v}$ as follows : We start by lifting each edge $e = vv'$ of $\mathcal{T}$ to an edge of the tiling in the plane, whose vertices say, are z_v and $z_{v'}$. Then we consider the segment whose vertices are $z_v + h_v$ and $z_{v'} + h_{v'}$ modulo Γ. For $\|\mathbf{h}\|$ small enough we obtain a tiling of $\mathbb{R}^2/\Gamma$ which we denote by $\mathcal{T}_{\mathbf{h}}$ for which the resultants

$$\mathcal{R}_v(\mathbf{h}) = \sum_{e \in \mathcal{E}_v} \frac{z_{v'} + h_{v'} - z_v - h_v}{|z_{v'} + h_{v'} - z_v - h_v|}$$

depend only on the vertices v. We note that $\mathcal{T}_{\mathbf{0}} = \mathcal{T}$ and that $\mathcal{R}_v(\mathbf{0}) = \mathcal{R}_v$.

Definition 1 *A tiling is said to be orientable if we can label its faces with $+$ and $-$ signs so that adjacent faces admit opposite signs.*

We note that if $\mathcal{T}$ is orientable then each vertex of $\mathcal{T}$ is incident with an even number of edges. If $\mathcal{T}$ is orientable, an orientation of $\mathcal{T}$ induces an orientation on the edges in the following sense : one assigns to an edge e in the boundary ∂f of a face f an initial vertex, denoted $init(e)$, and a terminal vertex denoted $ter(e)$, by tracing ∂f in the direct sense if f admits a + sign and in the indirect sense otherwise. We write $\vec{e} = \overrightarrow{vv'}$ to denote that $init(e) = v$ and $ter(e) = v'$. Once an orientation is fixed on an orientable tiling $\mathcal{T}$, we let $\sigma(f)$ denote the sign associated to f.

To each edge e of an oriented tiling we associate the complex number z_e as follows : we lift e to an edge of the tiling in the plane at some lift $z_{init(e)}$ of $init(e)$ and we set $z_e = z_{ter(e)} - z_{init(e)}$. Then the resultant at a vertex v can be written as

$$\mathcal{R}_v = \sum_{ter(e)=v} \frac{z_e}{|z_e|} - \sum_{init(e)=v} \frac{z_e}{|z_e|}.$$

Similarly for the tiling $\mathcal{T}_{\mathbf{h}}$, the resultants can be written as

$$\mathcal{R}_v(\mathbf{h}) = \sum_{ter(e)=v} \frac{z_e + h_e}{|z_e + h_e|} - \sum_{init(e)=v} \frac{z_e + h_e}{|z_e + h_e|},$$

where $h_e = h_{ter(e)} - h_{init(e)}$ and $\mathbf{h} = (h_v)_v$. We let $\mathcal{R}_{\mathcal{T}}(\mathbf{h}) = (\mathcal{R}_v(\mathbf{h}))_v$.

Definition 2 *The tiling $\mathcal{T}$ is said to be balanced if the resultant at each vertex of $\mathcal{T}$ is zero, i.e., $\mathcal{R}_{\mathcal{T}} = (\mathcal{R}_v)_v = 0$.*

We fix a vertex v of the tiling in the plane. For $i \in \{1, 2\}$ we denote by v_i the translate of v by T_i, and we choose a path $\mathcal{P}_i$ of consecutive edges of $\mathcal{T}$ which joins v to v_i. Let $\mathcal{B}_{T_i}$ denote the homology class of $\mathcal{P}_i$ in the quotient, say that an edge $e \in \mathcal{B}_{T_i}$ if e can be lifted to an edge of the path $\mathcal{P}_i$.

In what follows we assume that $\mathcal{T}$ is an oriented tiling. We associate to each $f \in \mathcal{F}$ and to each $\mathcal{B}_{T_i}$ a linear form on $\mathbb{C}^{n_e}$ as follows :

$$\phi_f(\mathbf{x}) = \sum_{e \in \partial f} x_e, \qquad \phi_{\mathcal{B}_{T_i}}(\mathbf{x}) = \sum_{e \in \mathcal{B}_{T_i}} x_e,$$

where $\mathbf{x} = (x_e)_e \in \mathbb{C}^{n_e}$. Then we consider the following subspace of $\mathbb{C}^{n_e}$,

$$\mathcal{W} = \big(\cap_f \ker \phi_f \big) \cap \big(\cap_i \ker \phi_{\mathcal{B}_{T_i}} \big).$$

We note that $\sum_{f \in \mathcal{F}} \sigma(f) \phi_f = 0$, so that $\mathcal{W}$ is defined by $(n_f - 1) + 2 = n_f + 1$ equations and

$$\dim_{\mathbb{C}} \mathcal{W} \geq n_e - (n_f + 1) = n_v - 1.$$

Remark 15 *The dimension of $\mathcal{W}$ is in fact equal to $n_v - 1$. This follows from the fact that the set of cycles formed by ∂f, $f \in \mathcal{F}$, and $\mathcal{B}_{T_i}$, $1 \leq i \leq 2$, generate the* *cycle space $\mathcal{C}(\mathcal{T})$ of the multi-graph given by the quotient tiling $\mathcal{T}$, whose dimension is given by the number of edges and the number of vertices : $n_e - (n_v - 1) = n_f + 1$ (see chapter 1 in [9]).*

One way to compute the dimension of $\mathcal{W}$ is the following : we write $\mathcal{W}$ as the solution space of a system of linear equations $M.\mathbf{z} = \mathbf{0}$, where $\mathbf{z} = (z_e)_e \in \mathbb{C}^{n_e}$ and M is a $(n_f + 2) \times n_e$ matrix, and we look at its rank. If we number the edges of $\mathcal{T}$ from 1 to n_e and the faces from 1 to n_f then $M = (m_{ij})$ is the matrix defined as follows : For $1 \leq i \leq n_f$, $m_{ij} = 1$ if $e_j \in \partial f_i$, and for $n_f + 1 \leq i \leq n_f + 2$, $m_{ij} = 1$ if $e_j \in \mathcal{B}_{T_{i-n_f}}$. All other entries of M are 0.

M is the transpose of the matrix of the cycles ∂f's, and $\mathcal{B}_{T_i}$'s, in the canonical basis of $\mathcal{E}(\mathcal{T})$ (see [9] for more details). Since the noted cycles generate the space $\mathcal{C}(\mathcal{T})$ whose dimension is $n_f + 1$, the rank of M is then equal to $n_f + 1$. The dimension of $\mathcal{W}$, the kernel of M, is then $n_v - 1$.

We also consider the subspace of $\mathbb{C}^{n_v}$

$$\mathcal{W}_v = \{(x_v)_v : \sum_v x_v = 0\},$$

whose dimension $\dim_{\mathbb{C}} \mathcal{W}_v = n_v - 1$. However, $\mathcal{W}$ and $\mathcal{W}_v$ are canonically isomorphic and will be identified.

Proposition 8 *Let $\mathcal{T}$ be an oriented tiling. The transformation which sends $(z_v)_v \in \mathbb{C}^{n_v}$ to $(z_e)_e \in \mathbb{C}^{n_e}$ with $z_e = z_{v'} - z_v$ for $\vec{e} = \overrightarrow{vv'}$, is an isomorphism from $\mathcal{W}_v \subset \mathbb{C}^{n_v}$ onto $\mathcal{W} \subset \mathbb{C}^{n_e}$.*

Proof. The fact that ∂f's and $\mathcal{B}_{T_i}$'s are cycles formed of consecutive edges of $\mathcal{T}$ implies that the image of $\mathcal{W}_v$ by the noted transformation is contained in $\mathcal{W}$. Moreover, the transformation is easily seen to be injective on $\mathcal{W}_v$. Since both spaces admit the same dimension the transformation is an isomorphism. □

Definition 3 *The tiling $\mathcal{T}$ is said to be rigid if $D\mathcal{R}_{\mathcal{T}}(\boldsymbol{0}) : \mathcal{W}_v \to \mathcal{W}_v$ is an isomorphism.*

The rigidity, as well as the orientability, of $\mathcal{T}$ depend on the period Γ. To see this, consider the regular tiling of the plane with unit squares and take $T_1 = (3, 0)$ and $T_2 = (0, 3)$. Then $\mathcal{T}$ is clearly orientable, and as we will see not rigid. On the other hand, if we take $T_1 = (1, 3)$ and $T_2 = (3, 2)$ then $\mathcal{T}$ is to the contrary rigid and not orientable. In fact the tiling by unit squares is orientable if and only if T_1 and T_2 are both of the form (n, m) with $n + m$ an even. We next prove general criteria for rigidity.

3.2.2 Rigid tilings

Roughly speaking, the tiling $\mathcal{T}$ is rigid if it admits no infinitesimal deformations which preserve the resultants at the corresponding vertices. In this section we provide

(a) Rigid (b) Non rigid

FIG. 3.1 – Examples of tilings

some criteria to decide the rigidity of a given tiling.

Theorem 10 *A tiling $\mathcal{T}$ all of whose faces are triangles is rigid, regardless of the period* Γ.

Before we proceed with the proof of this theorem we make the following observations. For $\mathbf{h} = (h_v)_v \in \mathbb{C}^{n_v}$ set

$$\ell(\mathbf{h}) = \sum_{e \in \mathcal{E}} \ell_e(\mathbf{h})$$

where

$$\begin{aligned} \ell_e(\mathbf{h}) &= |z_{v'} + h_{v'} - z_v - h_v| \\ &= |z_e + h_e| \end{aligned}$$

for $e = vv'$.

The functional ℓ is the sum of the lengths of the edges of $\mathcal{T}_{\mathbf{h}}$, and it is clear that ℓ is differentiable, in the real sense, in a neighborhood of $\mathbf{h} = \mathbf{0}$. We recall that if $f(x) = \|x\|$, $x \in \mathbb{R}^2 - \{0\}$, then

$$Df(x)h = \left\langle \frac{x}{\|x\|}, h \right\rangle$$

and

$$D^2 f(x)hh = \frac{1}{\|x\|^3}(\|x\|^2 \|h\|^2 - \langle x, h \rangle^2)$$

for all $h \in \mathbb{R}^2$. Note that $D^2 f(x)hh \geq 0$ and $D^2 f(x)hh = 0$ if and only if x and h are parallel.

Then for $e = vv'$, the total differential of ℓ_e at $\mathbf{h}$

$$D\ell_e(\mathbf{h})\mathbf{k} = D_{h_v}\ell_e(\mathbf{h})k_v + D_{h_{v'}}\ell_e(\mathbf{h})k_{v'},$$

where the partial differential of ℓ_e with respect to h_v, at $\mathbf{h} = (h_v)_v$, is calculated usinf by the above formula for f to be

$$D_{h_v}\ell_e(\mathbf{h})k_v = \left\langle \frac{z_v + h_v - z_{v'} - h_{v'}}{|z_{v'} + h_{v'} - z_v - h_v|}, k_v \right\rangle,$$

and the total differential of ℓ at $\mathbf{h}$ is then

$$\begin{aligned} D\ell(\mathbf{h})\mathbf{k} =& \sum_e D\ell_e(\mathbf{h})\mathbf{k} \\ =& \sum_e D_{h_v}\ell_e(\mathbf{h})k_v + D_{h_{v'}}\ell_e(\mathbf{h})k_{v'} \\ =& \frac{1}{2}\sum_v \sum_{e\in\mathcal{E}_v} D_{h_v}\ell_e(\mathbf{h})k_v + \frac{1}{2}\sum_{v'} \sum_{e\in\mathcal{E}_{v'}} D_{h_{v'}}\ell_e(\mathbf{h})k_{v'} \\ =& \sum_v \langle \mathcal{R}_v(\mathbf{h}), k_v\rangle \end{aligned}$$

In particular, we can see that $D\ell(0) = 0$ if and only if $\mathcal{R}_{\mathcal{T}} = \mathbf{0}$ which implies that balanced tilings are critical points for the total-edge-length functional.

We differentiate the equation

$$D\ell(\mathbf{h})\mathbf{k} = \sum_v \langle \mathcal{R}_v(\mathbf{h}), k_v\rangle$$

to obtain

$$D^2\ell(\mathbf{0})\mathbf{hh} = \sum_v \langle D\mathcal{R}_v(\mathbf{0})\mathbf{h}, h_v\rangle.$$

Thus

$$D\mathcal{R}_{\mathcal{T}}(\mathbf{0})\mathbf{h} = \mathbf{0} \text{ only if } D^2\ell(\mathbf{0})\mathbf{hh} = 0.$$

On the other hand,

$$D^2\ell(\mathbf{0})\mathbf{hh} = \sum_e D^2\ell_e(\mathbf{0})\mathbf{hh}$$

with

$$D^2\ell_e(\mathbf{0})\mathbf{hh} = D^2_{h_v h_v}\ell_e(\mathbf{0})h_v h_v + 2D^2_{h_v h_{v'}}\ell_e(\mathbf{0})h_v h_{v'} + D^2_{h_{v'} h_{v'}}\ell_e(\mathbf{0})h_{v'} h_{v'},$$

for $e = vv'$. We calculate these second derivatives using the formula for $D^2 f$ given above and we obtain

$$D^2\ell(\mathbf{0})\mathbf{hh} = \sum_e \frac{1}{|z_e|^3}\big(|z_e|^2 |h_e|^2 - \langle z_e, h_e\rangle^2 \big).$$

Therefore $D^2\ell(\mathbf{0})\mathbf{hh} = 0$ is equivalent to that fact that for each edge $e = vv'$, z_e and $z_e + h_e$ are parallel. We have just proved

Proposition 9 *Let $\mathcal{T}$ be a tiling such that $D\mathcal{R}_{\mathcal{T}}(\boldsymbol{0})(\boldsymbol{h}) = \boldsymbol{0}$ for some $\boldsymbol{h} \in \mathbb{C}^{n_v}$. Then the edges of the tiling $\mathcal{T}_{\mathbf{h}}$, obtained by deforming $\mathcal{T}$ by $\boldsymbol{h}$, are parallel to the corresponding edges of $\mathcal{T}$.*

Now we turn to the proof of the theorem which follows readily from the established facts. Suppose that all the faces of $\mathcal{T}$ are triangles and that $D\mathcal{R}_{\mathcal{T}}(\mathbf{h}) = \mathbf{0}$ for some $\mathbf{h} \in \mathbb{C}^{n_v}$. The conclusion above implies that the faces of $\mathcal{T}_{\mathbf{h}}$ are also triangles which are homothetic to the corresponding ones of $\mathcal{T}$. The connectedness of the tiling implies that we have the same homothety factor for all triangles. Since $\mathbf{h} \in \mathcal{W}_v = \mathcal{W}$ then $\mathcal{T}$ and $\mathcal{T}_{\mathbf{h}}$ have the same period, which implies that the homothety factor is 1 and that $h_v = h_{v'}$ for any two vertices of $\mathcal{T}$. Hence $\mathcal{T}_{\mathbf{h}}$ is a translate of $\mathcal{T}$ and since $\sum_v h_v = 0$ we get that $\mathbf{h} = \mathbf{0}$, which proves the theorem.

In fact we do not need all the faces to be triangles. The above proof adapts to the following situation

Theorem 11 *Let $\mathcal{T}$ be a tiling in which every edge bounds at least a triangle and that any two triangles in $\mathcal{T}$ can be connected by a path of adjacent triangles. Then $\mathcal{T}$ is rigid.*

We end this section by giving a criterion for the rigidity of regular tilings by units squares.

Theorem 12 *Let $\mathcal{T}$ be the regular tiling of the plane by unit squares whose periods are taken to be $T_1 = (n_1, m_1)$ and $T_2 = (n_2, m_2)$. Then $\mathcal{T}$ is rigid if and only if (n_1, n_2) and (m_1, m_2) are couples of relatively prime numbers.*

Proof. We remark that (m_1, m_2) (resp. (n_1, n_2)) is a couple of relatively prime numbers if and only if the horizontal (resp. vertical) lines of the tiling project into a single horizontal (resp. vertical) line in the quotient $\mathbb{R}^2/\Gamma$. If we have more than one horizontal (resp. vertical) line in the quotient it is easy to see that we can deform the tiling by displacing one of the horizontal (resp. vertical) lines while keeping the others fixed (to displace a line we may displace its lift in $\mathbb{R}^2$ then consider the displaced line in the quotient). This implies that the condition (n_1, n_2) and (m_1, m_2)

are couples of relatively prime numbers is necessary.

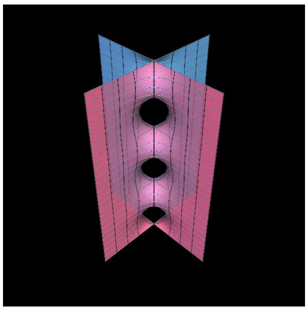
Reciprocally, we suppose that $D\mathcal{R}_{\mathcal{T}}(\mathbf{0})\mathbf{h} = 0$ for a tiling whose horizontal (resp. vertical) edges lie on the same horizontal (resp. vertical) line, with $\mathbf{h} \in \mathcal{W}_v$.
We choose a vertex $v \in \mathcal{V}$ and we translate $\mathcal{T}_{\mathbf{h}}$ back to v by $h_v(1,..,1) \in \mathbb{C}^{n_v}$. We notice that the translations of $\mathcal{T}$, which correspond to vectors of $\mathbb{C}^{n_v}$ with equal components, are in the kernel of $D\mathcal{R}_{\mathcal{T}}(\mathbf{0})$. Thus we obtain a deformation of $\mathcal{T}$ by $\mathbf{h} - h_v(1,..,1) \in \mathbb{C}^{n_v}$ which is in the kernel of $D\mathcal{R}_{\mathcal{T}}(\mathbf{0})$. Proposition 3 then implies that we obtain a tiling whose horizontal (resp. vertical) edges lie on the horizontal (resp. vertical) line of $\mathcal{T}$ through v, which implies that the two tilings coincide. Hence, $\mathbf{h} = h_v(1,..,1)$ and as $\mathbf{h} \in \mathcal{W}_v$ we conclude that $\mathbf{h} = 0$.

3.3 Main Result

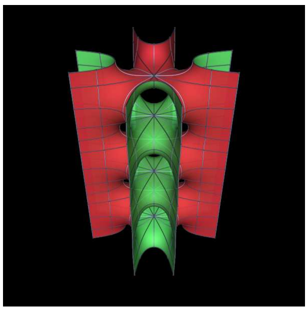
In this section we present the main result of the paper. We first proceed with a reminder of some properties of the Karcher saddle towers.

3.3.1 Karcher Saddle Towers

These are simply periodic minimal surfaces with $2k$ Scherk type ends, $k \geq 2$, which can be constructed by taking conjugates of Jenkins-Serrin graphs over equilateral convex $2k$-gons, and completing them by symmetries, see [17]. For $k = 2$ these surfaces are also known as the simply periodic Scherk surfaces.



(a) Scherk singly periodic



(b) A Karcher saddle tower with eight ends

FIG. 3.2 – Karcher saddle towers

In fact, for any set $\{v_1, ..., v_{2k}\}$ of distinct horizontal vectors of equal length, $k \geq 2$, such that $\sum v_i = 0$ there exists such a surface with $2k$ ends each of which is parallel to some vector v_i. This surface can be constructed as follows : The set of vectors $\{v_i\}$ noted above defines a strictly convex equilateral $2k$-gon whose edges we alternately label $+\infty$ and $-\infty$. As the polygon is equilateral (and non-special in the sense of

[27]), the existence of a minimal graph over its interior and taking the values $+\infty$ and $-\infty$ on the corresponding boundary edges is guaranteed by the Jenkins-Serrin theorem (see [16]). The graph is a conformal disc bounded by vertical lines over the vertices of the polygon. Therefore, the conjugate surface of such a Jenkins-Serrin graph is bounded by horizontal arcs which lie alternately in two horizontal planes. The conjugate surface, which is a graph (by a theorem of R. Krust), meets these planes perpendicularly and can be extended by reflection to a complete Karcher saddle tower. The intrinsic distance between adjacent vertical lines is the same as between the corresponding planar symmetry lines. Since a Jenkins-Serrin graph has a limiting normal as we approach the edges (which is a horizontal vector orthogonal to the corresponding edge), the intrinsic distance between adjacent vertical lines is equal to the length of the edges of the polygon. Therefore, the vertical period of the Karcher saddle tower is twice the length of the edges of the polygon.

The normals at the ends of the Jenkins-Serrin graph and its conjuagte are the same, so a rotation about the vertical axis of the Karcher saddle tower by $\frac{\pi}{2}$ implies that the ends are in the direction of the edges v_i. We note that the only horizontal normals of a Jenkins-Serrin graph are along the vertical boundary lines, otherwise the surface could not be a graph. This implies that the only horizontal normals to the Karcher saddle tower lie on the lines of planar symmetry and along these the normal rotates monotonely, from one limiting normal on one edge to the limiting normal on an adjacent edge. Hence, the length of arc of great circle, described by the Gauss map along a horizontal symmetry line of the Karcher saddle tower (which is the conjugate of the vertical line bounding the Jenkins-Serrin graph over the meeting point A_i of v_i and v_{i+1}) is $\pi - a_i$, where a_i is the exterior angle of the polygon at A_i. This implies that

$$\sum_i \pi - a_i = 2\pi d$$

where d is the degree of the Gauss map. Since the sum of exterior angles of our polygon is 2π, then the above equation becomes

$$2k\pi - 2\pi = 2\pi d$$

and the degree of the Gauss map is therefore $k-1$.

If we quotient the Karcher saddle tower by its smallest period we obtain a conformal sphere with $2k$ punctures and then we can see that we may assume the lines of planar symmetry and the punctures to lie on the real line. In other words, if $\Phi_0 = (\phi_1^0, \phi_2^0, \phi_3^0)$ denotes the Weierstrass data over a punctured sphere of the a Karcher saddle tower, then we may assume that the poles of Φ^0 lie on the real line which is a line of planar symmetry. As the zeros of ϕ_1^0 are realized at points where the Gauss map is horizontal, then we may assume these zeros to be real.

The main result of the paper is the following

Theorem 13 *Let $\mathcal{T}$ be a balanced tiling in the plane which is invariant by two independent translations T_1 and T_2. Let Γ denote the group generated by T_1 and T_2, and denote by $\mathcal{T}$ the corresponding quotient tiling in $\mathbb{R}^2/\Gamma$.*

If the quotient tiling $\mathcal{T}$ is orientable and rigid then for any $\varepsilon \neq 0$ sufficiently small, there exists an embedded triply periodic minimal surface M_ε with horizontal period Γ and a vertical period $(0, 0, 2\pi\varepsilon^2)$ such that :

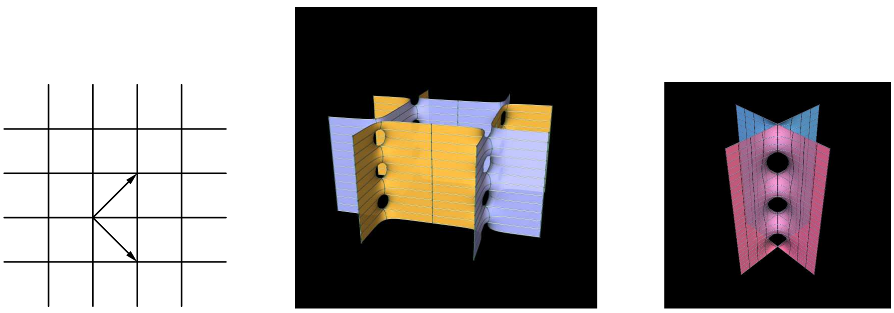
1. *M_ε is symmetric with respect to the horizontal plane and depends continuously on ε.*

2. *When $\varepsilon \to 0$, M_ε converges, on compact subsets of $\mathbb{R}^3$ and for the ambient metric, to the set $\mathcal{T} \times \mathbb{R}$.*

3. *In a neighborhood of each vertex v of $\mathcal{T}$ in the plane, when scaled by ε^{-2}, M_ε looks like a Karcher saddle tower M_v whose period is equal to $(0, 0, 2\pi)$, and whose ends are in a one-to-one correspondence with the edges ending at v in such a way that, the asymptotic vertical half-plane to an end of M_v is parallel to its corresponding edge and points in the same direction.*

   More precisely, for each v there exists a horizontal vector ν_ε such that, $\varepsilon^{-2}(M_\varepsilon - \nu_\varepsilon)$ converges to M_v on compact subsets of $\mathbb{R}^3$.

Example Consider the tiling of $\mathbb{R}^2$ by unit squares, and let $T_1 = (1, 1)$ and $T_2 = (-1, 1)$. In $\mathbb{R}^2/\Gamma$, $\mathcal{T}$ has two vertices and two faces. Each M_v is a singly periodic Scherk surface.



(a) A tiling $\mathcal{T}$ (b) A member of the family M_ε (c) The same M_v for each v

Remark 16 *If one thinks of M_ε as gluing, for each $e = vv'$, M_v and $M_{v'}$ along their ends parallel to e, then M_ε would depend on the signs one associates to the vertices and faces of the tiling in the sense explained in the next section and as illustrated by examples 1 and 2.*

3.3.2 Examples

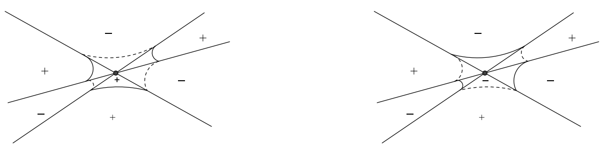
One can apply Theorem 13 to carefully tailored tilings as to recover many of the (deformed) classical examples of triply periodic minimal surfaces. For a given tiling, the family of surfaces resulting from Theorem 13 depend on how one "places" the M_v's as it is shown in the examples below. In fact, the level curves of a Karcher saddle tower in a plane of symmetry, say the plane $x_3 = 0$, can be arranged about their asymptotic lines in exactly two ways, as demonstrated in the figure below. The dotted lines represent the level curves in the symmetry plane $x_3 = -\pi$.

(d) Level curves of a given saddle tower

(e) Level curves of the same saddle tower translated by half its vertical period

FIG. 3.3 – A singly periodic Sherk surface

We consider a tiling $\mathcal{T}$ as in Theorem 13 on which we fix an orientation. We fix a sign on each of the vertices of $\mathcal{T}$ and at a vertex v we place M_v as demonstrated in the figure below, which without loss of generality shows the placement for a saddle tower with six ends.



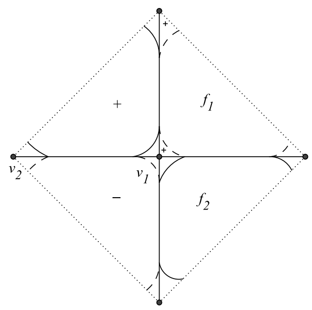
(a) Level curves a given saddle tower

(b) Level curves of the same saddle tower translated by half its vertical period

FIG. 3.4 – The positioning of M_v at a vertex v of $\mathcal{T}$ depending on the sign of faces surrounding it

Therefore, for $v \in \mathcal{V}$, one may place M_v at v in two ways and when one glues M_v and $M_{v'}$ along their ends pointing along $e = vv'$, the obtained surface depends on how each of the surfaces was placed at v and v' in the first place. One can think that for $e = vv'$, M_v and $M_{v'}$ are both placed following the same rule, as explained in the figure above, if $\sigma(e) > 0$ and following opposite rules otherwise.

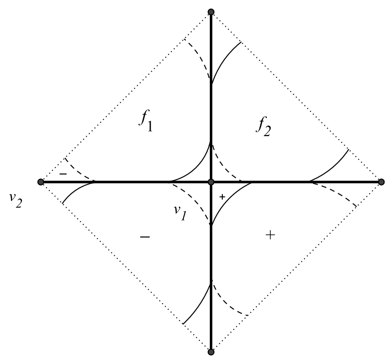
Example 1. The following configuration corresponds to a member of the P-surface family. The dots represent the vertices of the tiling and the solid lines its edges. Then $n_v = 2$, $n_e = 4$ and $n_f = 2$ which implies that in the quotient the genus of the obtained surface is equal to three.



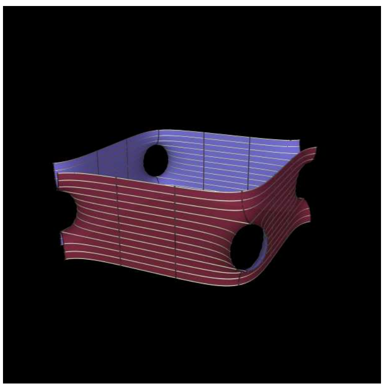
(a) P configurtaion

(b) A fundamental piece of a deformed P-surface

Example 2 The following is the CLP surface.

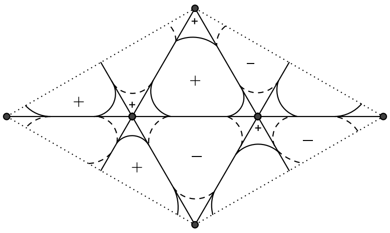


(c) CLP configuration

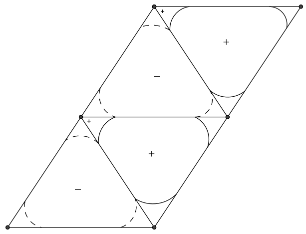


(d) A fundamental piece of a deformed CLP-surface

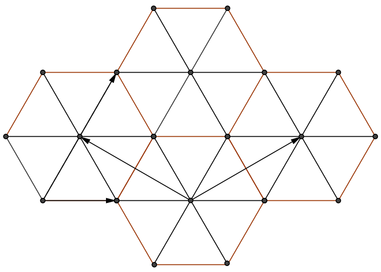
Example 3 The following is the H surface.



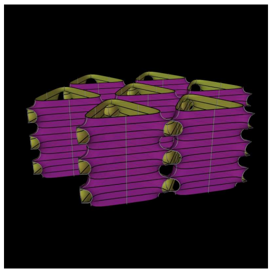
(e) One possible H configuration



(f) Another H configuration depending on the choice of period of the plane tiling



(g) The plane tiling with the two periods giving the above configurations shown



(h) A deformed H-surface in a box

Example 4 The following is the H'-T surface.

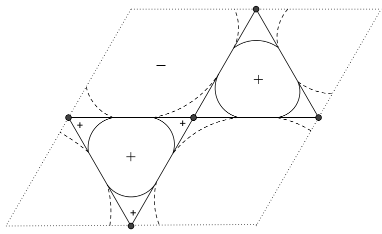
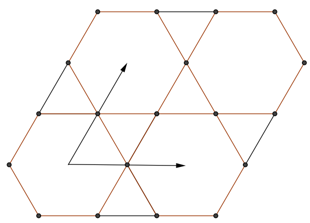
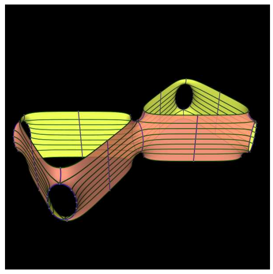


FIG. 3.5 – H'-T configuration



(a) The tiling in the plane



(b) A deformed H'T-surface fundamental piece

3.4 The setting of the construction

One may ask what Riemann surfaces are candidates for the construction ? To answer this question we first recall that conformally the Karcher surfaces are punctured spheres. Then we note that once the gluing is performed as explained in the introduction, in the quotient of $\mathbb{R}^3$ by the lattice L_ε , our surfaces would look like n_v sphere connected with n_e necks. In the quotient, the necks correspond to the glued wings and will be along the edges.

These observations inspire us to define the Riemann surfaces underlying our construction by opening the nodes of a Riemann surface with nodes. The conformal types of these surfaces are to be fixed by adjusting some parameters related to their construction, in such a way that the Weierstrass data to be furnished, will define conformal minimal immersions whose periods lie in the lattice L_ε. The Weierstrass data on the initial Riemann surface with nodes will give Karcher saddle towers on each its component punctured spheres. We refer the reader to [15] and [20] for more details on Riemann surfaces with nodes and the degeneration of Riemann surfaces. For the techniques implied in this and later parts of the paper, we refer the reader to [34] and [35].

3.4.1 A degenerate Riemann Surface

For each vertex v of the quotient tiling $\mathcal{T}$ we consider a copy of the sphere $\mathbb{C}\cup\{\infty\}$, which we denote by $\widehat{\mathbb{C}}_v$. We fix $d(v)$ points p_1^v, $p_2^v, \ldots,$ $p_{d(v)}^v$ in $\widehat{\mathbb{C}}_v$ taken on the real axis. We assume that these points form an increasing sequence and we put these points in a one-to-one correspondence with the edges ending at v as follows : we number the edges ending at v in the direct sense then associate the point p_i^v with the edge numbered i. In what follows we denote by p_e^v the point of $\widehat{\mathbb{C}}_v$ which corresponds to e. We let $\mathbf{p} = (p_e^v)_{\substack{v\in\mathcal{V}\\ e\in\mathcal{E}_v}}$.

For each edge $e = vv'$ we identify the points of $\widehat{\mathbb{C}}_v$ and $\widehat{\mathbb{C}}_{v'}$ which correspond to e, namely, $p_e^v \in \widehat{\mathbb{C}}_v$ and $p_e^{v'} \in \widehat{\mathbb{C}}_{v'}$. We refer to the point which results from this identification as the node at e. Thus we obtain a Riemann surface with nodes, *i.e.*, a connected complex space in which every point has a neighborhood isomorphic to either $|z| < 1$ in $\mathbb{C}$ or $|z| < 1$, $|w| < 1$, $z.w = 0$ in $\mathbb{C}^2$.

3.4.2 Opening the nodes

For our purposes we open the node at $e = vv'$ as follows : Let

$$d = \min_{e \neq \tilde{e}} \{dist(p_e^v, p_{\tilde{e}}^v), dist(p_e^{v'}, p_{\tilde{e}}^{v'})\}$$

and consider the local coordinates in $\widehat{\mathbb{C}}_v$ and $\widehat{\mathbb{C}}_{v'}$, $z_e^v = z - p_e^v$ in a neighborhood of p_e^v and $z_e^{v'} = z - p_e^{v'}$ in a neighborhood of $p_e^{v'}$. Choose $0 < \epsilon < \dfrac{d}{2}$ and a real parameter t_e such that $0 < |t_e| < \epsilon^2$. We require t_e to be positive if $\sigma(e) = -1$ and negative otherwise ($\sigma(e)$ is as in Proposition 7). We remove the discs

$$\mathcal{D}_e^v = \{|z_e^v| < \frac{|t_e|}{\epsilon}\} \text{ and } \mathcal{D}_e^{v'} = \{\left|z_e^{v'}\right| < \frac{|t_e|}{\epsilon}\},$$

and we identify the points of the annuli

$$U_e^v = \{|t_e| \le |z_e^v| \le \epsilon\} \text{ and } U_e^{v'} = \{|t_e| \le \left|z_e^{v'}\right| \le \epsilon\},$$

by the mapping

$$z_e^v.z_e^{v'} = t_e.$$

Let $\mathbf{t} = (t_e)_{e\in\mathcal{E}}$, and consider the neighborhoods of $\mathbf{0} \in \mathbb{R}^{n_e}$ $\mathcal{N}(\mathbf{0}) = \{\mathbf{t} | \, |t_e| < \epsilon^2\}$ and $\mathcal{N}^*(\mathbf{0}) = \{\mathbf{t} | \forall e \in \mathcal{E}, \ 0 < |t_e| < \epsilon^2\}$. By opening the n_e nodes for $\mathbf{t} \in \mathcal{N}^*(\mathbf{0})$ of the Riemann surface with nodes described above, we obtain a regular Riemann surface $\Sigma_{\mathbf{t},\mathbf{p}}$ which depends holomorphically on the parameters $\mathbf{t} = (t_e)_e$ and the points $\mathbf{p} = (p_e^v)_{\substack{v\in\mathcal{V} \\ e\in\mathcal{E}_v}}$ and whose genus is $g = n_e - (n_v - 1) = n_f + 1$.

We notice that if $t_e = 0$ for some $e = vv' \in \mathcal{E}$, the procedure of opening a node, in fact, produces a node by identifying p_e^v and $p_e^{v'}$, so that for $\mathbf{t} \in \mathcal{N}(\mathbf{0})$ one would possibly obtain Riemann surfaces with nodes. When $\mathbf{t} = \mathbf{0}$ we obtain $\Sigma_{\mathbf{0},\mathbf{p}}$ the Riemann surface with nodes defined above with a node at each e, and we say that $\Sigma_{\mathbf{t},\mathbf{p}}$ degenerates into $\Sigma_{\mathbf{0},\mathbf{p}}$. In our work, we will be interested only in the surfaces obtained by opening the nodes for $\mathbf{t} \in \mathcal{N}^*(\mathbf{0})$.

Remark 17 *Since the points p_e^v and the parameters t_e are real, complex conjugation is well defined in $\Sigma_{\mathbf{t},\mathbf{p}}$, $\mathbf{t} \in \mathcal{N}^*(\mathbf{0})$.*

3.4.3 A generator set for the homology group of $\Sigma_{\mathbf{t},\mathbf{p}}$

We fix an orientation on $\mathcal{T}$ and we introduce on $\Sigma_{\mathbf{t},\mathbf{p}}$, $\mathbf{t} \in \mathcal{N}^*(\mathbf{0})$, a system of cycles $\{A_e\}_e$, $\{B_f\}_f$, $\{B_{T_1}, B_{T_2}\}$ defined up to homology, which generates the respective homology group of the underlying surface.

1. For each edge $\vec{e} = \overrightarrow{vv'}$, we let A_e denote the positively oriented circle $C_\epsilon(p_e^v)$.
2. For each face of $\mathcal{T}$ we define a corresponding cycle B_f. Let $\vec{e} = \overrightarrow{vv'}$ be an edge of $f \in \mathcal{F}$ and let $e' \in \mathcal{E}_f$ be such that $ter(e') = init(e)$. We associate a composition of oriented curves to e, denoted b_e, as described below.

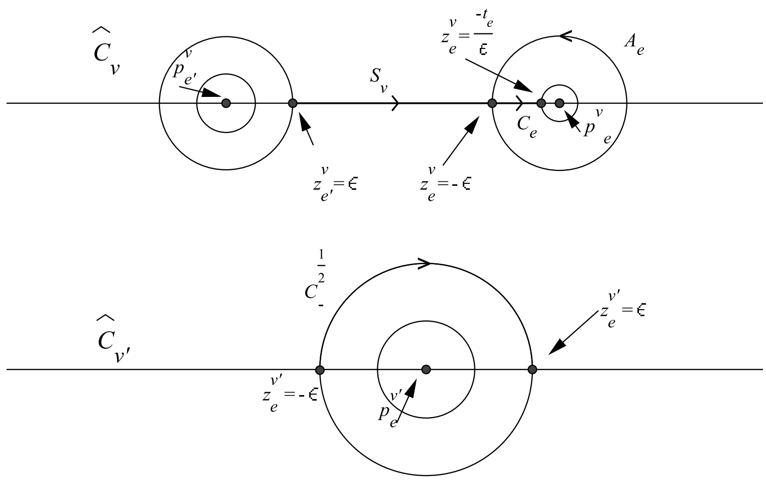


FIG. 3.6 – A curve b_e for $\sigma(f) = -1$ and $\sigma(e) = -1$

(a) If $\sigma(f) = -1$ we remark that $p_{e'}^v < p_e^v$ with no other points $p_{\tilde{e}}^v$ in between (except when $p_{e'}^v = p_{d(v)}^v$ and $p_e^v = p_1^v$ but still we think of p_1^v as succeeding $p_{d(v)}^v$. One starts tracing the real line at $p_{d(v)}^v$ till the point at ∞ in $\widehat{\mathbb{C}}_v$, and one continues till p_1^v !). We proceed as follows :

i. If $\sigma(e) = +1$, in which case $t_e < 0$ and the point $z_e^v = \frac{t_e}{\epsilon}$ is identified with the point $z_e^{v'} = \epsilon$, we consider :

A. The real segment $\mathcal{S}_{init(e)}$ in $\widehat{\mathbb{C}}_v$ which goes from the point $z_{e'}^v = \epsilon$ to the point $z_e^v = -\epsilon$.

B. The real segment $\mathcal{C}_e$ in $\widehat{\mathbb{C}}_v$ which goes from the point $z_e^v = -\epsilon$ to the point $z_e^v = \frac{-|t_e|}{\epsilon}$.

In the case $p_{e'}^v = p_{d(v)}^v$ and $p_e^v = p_1^v$, $\mathcal{S}_{init(e)}$ goes from $z_{e'}^v = \epsilon$ to the point $z_e^v = -\epsilon$ passing through the point at ∞.

ii. If $\sigma(e) = -1$, in which case $t_e > 0$ and the point $z_e^v = \frac{-t_e}{\epsilon}$ is identified with the point $z_e^{v'} = -\epsilon$, we consider

A. The real segment $\mathcal{S}_{init(e)}$ in $\widehat{\mathbb{C}}_v$ as defined above.

B. The real segment $\mathcal{C}_e$ in $\widehat{\mathbb{C}}_v$ also as defined above.

C. The negatively oriented semi-circle $C_-^{\frac{1}{2}}(p_e^{v'}) = \{\left|z_e^{v'}\right| = \epsilon : Re(z_e^{v'}) > 0\}$

In this case, the semi circle $C_-^{\frac{1}{2}}(p_e^{v'})$ goes from the point $z_e^{v'} = -\epsilon$ to the point $z_e^{v'} = \epsilon$.

The curve b_e joins the point $z_{e'}^v = \epsilon$ and the point $z_e^{v'} = \epsilon$.

(b) If $\sigma(f) = +1$ we remark that $p_e^v < p_{e'}^v$ with no other points $p_{\tilde{e}}^v$ in between (the case , and we proceed as follows :

i. If $\sigma(e) = +1$, in which case $t_e < 0$ and the point $z_e^v = \frac{|t_e|}{\epsilon}$ is identified with the point $z_e^{v'} = -\epsilon$, we consider :

A. The real segment $\mathcal{S}_{init(e)}$ in $\widehat{\mathbb{C}}_v$ which goes from the point $z_{e'}^v = -\epsilon$ to the point $z_e^v = \epsilon$.

B. The real segment $\mathcal{C}_e$ in $\widehat{\mathbb{C}}_v$ which goes from the point $z_e^v = \epsilon$ to the point $z_e^v = \frac{|t_e|}{\epsilon}$.

ii. If $\sigma(e) = -1$, in which case $t_e > 0$ and the point $z_e^v = \frac{t_e}{\epsilon}$ is identified with the point $z_e^{v'} = \epsilon$, we consider

A. The real segment $\mathcal{S}_{init(e)}$ in $\widehat{\mathbb{C}}_v$ as defined above.

B. The real segment $\mathcal{C}_e$ in $\widehat{\mathbb{C}}_v$ also as defined above.

C. The positively oriented semi-circle $C_+^{\frac{1}{2}}(p_e^{v'}) = \{\left|z_e^{v'}\right| = \epsilon : Re(z_e^{v'}) > 0\}$

In this case, the semi circle $C_+^{\frac{1}{2}}(p_e^{v'})$ goes from the point $z_e^{v'} = \epsilon$ to the point $z_e^{v'} = -\epsilon$.

The curve b_e joins the point $z_{e'}^v = -\epsilon$ and the point $z_e^{v'} = -\epsilon$.

For $f \in \mathcal{F}$, we define B_f to be the oriented cycle which is the composition of the oriented curves $(b_e)_{e \in \mathcal{E}_f}$.

3. We fix $i \in \{1, 2\}$ and we associate to $\mathcal{B}_{T_i}$, a cycle B_{T_i} as follows. Consider a path $\mathcal{P}_i$ of consecutive edges of $\mathcal{T}$, joining a vertex $v_\circ$ to its translate by T_i.

In the quotient, $\mathcal{P}_i$ is a closed cycle formed by edges of $\mathcal{T}$. To each $v \in \mathcal{P}_i$ we associate a composition of curves, which we denote by b_i^v, defined as follows :

Let e and e' denote the consecutive edges of $\mathcal{P}_i$ which end at v. We number the edges in $\mathcal{E}_v$ in the direct sense, and we suppose that e takes the number m and e' the number n, $1 \leq m < n \leq d(v)$. We consider the $n - m$ faces around v bounded by the edges numbered m, $m + 1$,..., $n - 1$, n, and we let f_k denote such a face which is bounded by the edges numbered $m + k - 1$ and $m + k$, $1 \leq k \leq n - m$. In $\widehat{\mathbb{C}_v}$, we associate to a face f_k the curve $b_{i,k}^v$ defined as follows :

(a) If $\sigma(v).\sigma(f_k) > 0$ (v and f_k have the same sign) then $b^v_{i,k}$ is the real segment which goes from the point $z^v_{m+k-1} = \epsilon$ to the point $z^v_{m+k} = -\epsilon$.

(b) If $\sigma(v).\sigma(f_k) < 0$ then $b^v_{i,k}$ is the composition of the following curves :

i. The negatively oriented semi-circle $C_-^{\frac{1}{2}}(p^v_{m+k-1})$ which joins the points $z^v_{m+k-1} = -\epsilon$ to the point $z^v_{m+k-1} = \epsilon$.

ii. The real segment which goes from the point $z^v_{m+k-1} = \epsilon$ to the point $z^v_{m+k} = -\epsilon$.

iii. The positively oriented semi-circle $C_+^{\frac{1}{2}}(p^v_{m+k})$ which joins the points $z^v_{m+k} = -\epsilon$ to the point $z^v_{m+k-1} = \epsilon$.

We let b^v_i be the composition of all the curves $b^v_{i,k}$, $1 \leq k \leq n-m$. Now to each $e = vv'$ in $\mathcal{P}_i$ we associate a curve b^e_i defined as follows :

(a) If $\sigma(e) = +1$ then v and v' have the same sign, in which case we remark that if b^v_i ends at $z^v_e = \pm\epsilon$ then $b^{v'}_i$ starts at $z^{v'}_e = \mp\epsilon$ and vice versa. The fact that $t_e < 0$ implies that the point $z^v_e = \pm\epsilon$ is identified with $z^{v'}_e = \pm\frac{t_e}{\epsilon}$ and thus the curves b^v_i and $b^{v'}_i$ could be joined by the segment which goes from $z^{v'}_e = \pm\frac{t_e}{\epsilon}$ to $z^{v'}_e = \mp\epsilon$. We let b^e_i be this segment.

(b) If $\sigma(e) = -1$ then v and v' have opposite signs, in which case we remark that if b^v_i ends at $z^v_e = \pm\epsilon$ then $b^{v'}_i$ starts at $z^{v'}_e = \pm\epsilon$ and vice versa. The fact that $t_e > 0$ implies that the point $z^v_e = \pm\epsilon$ is identified with $z^{v'}_e = \pm\frac{t_e}{\epsilon}$ and thus the curves b^v_i and $b^{v'}_i$ could be joined by the segment which goes from $z^{v'}_e = \pm\frac{t_e}{\epsilon}$ to $z^{v'}_e = \pm\epsilon$. We let b^e_i be this segment.

The oriented cycle B_{T_i} is defined as the composition of all the oriented curves b^e_i, $e \in \mathcal{P}_i$, and the oriented curves b^v_i, v the vertex of $e \in \mathcal{P}_i$.

A cycle A_e, $e = vv'$, goes around the neck which joins $\widehat{\mathbb{C}_v}$ and $\widehat{\mathbb{C}_{v'}}$. A cycle B_f goes through each of the necks which correspond to $e \in \partial f$, and joins the necks which correspond to two consecutive edges $e, e' \in \partial f$ which end at v, by a curve in $\widehat{\mathbb{C}_v}$. Similarly, a representative cycle of B_{T_i} goes through the necks which correspond to $e \in \mathcal{P}_i$, and joins two such necks which correspond to two edges ending at a common vertex, by a curve in the sphere which corresponds to that vertex. It is clear, from the topological picture of $\Sigma_{\mathbf{t},\mathbf{p}}$, $\mathbf{t} \in \mathcal{N}^*(\mathbf{0})$, that the set of cycles $\{A_e\} \cup \{B_e\} \cup \{B_{T_i}\}$ generates the homology group of $\Sigma_{\mathbf{t},\mathbf{p}}$.

Remark 18 *By Proposition* 7 *each face $f \in \mathcal{F}$ has an even number of edges with a negative sign. Thus, as it follows from the definition, each cycle B_f contains an even number of semi-circles, each of which corresponds to an edge of f whose sign is* -1. *Similarly, each cycle B_{T_i} contains an even number of semi-circles, contributed by faces whose sign is -1, as seen in the definition of B_{T_i}.*

3.4.4 Regular differentials

In this section, following the lines of Masur [20], we extend the notion of holomorphic differentials to that of regular differentials on Riemann surfaces with nodes.

For our purposes, we restrict ourselves to the Riemann surfaces with nodes $\Sigma_{\mathbf{0},\mathbf{p}}$.

Definition 4 *A regular differential ω on $\Sigma_{\mathbf{0},\mathbf{p}}$ is a 1-form such that for each $v \in \mathcal{V}$, ω is holomorphic in $\widehat{\mathbb{C}}_v - \{p_e^v\}_{e\in\mathcal{E}_v}$ and ω admits a pole of order ≤ 1 at p_e^v, for each $e \in \mathcal{E}_v$. Moreover, for each $e \in \mathcal{E}$ with $e = vv'$, the residues of ω at p_e^v and $p_e^{v'}$ must be opposite.*

We let $\Omega^1(\Sigma_{\mathbf{0},\mathbf{p}})$ denote the space of regular 1-forms on $\Sigma_{\mathbf{0},\mathbf{p}}$. The space $\Omega^1(\Sigma_{\mathbf{0},\mathbf{p}})$ has dimension g, where g is the genus of $\Sigma_{\mathbf{t},\mathbf{p}}$ for $\mathbf{t} \in \mathcal{N}^*(\mathbf{0})$. In fact, more generally $\Sigma_{\mathbf{t},\mathbf{p}}$ can be constructed by opening the nodes of $\Sigma_{\mathbf{0},\mathbf{p}}$ for $\mathbf{t} \in \mathbb{C}^{n_e}$ near $\mathbf{0}$, and the space $\Omega^1(\Sigma_{\mathbf{t},\mathbf{p}})$ depends holomorphically on the parameters $\mathbf{t}$ in the following sense : Proposition 4.1 of [20] says that there exists a basis $\omega_{1,\mathbf{t}}, ..., \omega_{g,\mathbf{t}}$ of $\Omega^1(\Sigma_{\mathbf{t},\mathbf{p}})$, the space of holomorphic forms on $\Sigma_{\mathbf{t},\mathbf{p}}$, which depends holomorphically on $\mathbf{t}$ in a neighborhood of $\mathbf{0}$. For $\mathbf{t} = \mathbf{0}$ the forms $\omega_{i,\mathbf{t}}$ degenerate to a basis of regular differentials $\omega_{i,\mathbf{0}}$ on the Riemann surface with nodes $\Sigma_{\mathbf{0},\mathbf{p}}$. As we restrict ourselves to real parameters t_e, the dependence of $\Sigma_{\mathbf{t},\mathbf{p}}$ on $\mathbf{t}$ is real analytic. This fact is fundamental for the construction we have in mind as we'll apply the implicit function theorem at $\mathbf{t} = 0$, passing from regular differentials on Riemann surfaces with nodes to holomorphic differentials on Riemann surfaces.

For each $v \in \mathcal{V}$ we consider the domain $\mathcal{G}_v(\epsilon)$ of $\Sigma_{\mathbf{t},\mathbf{p}}$ which consists of the complement in $\widehat{\mathbb{C}}_v$ of the discs $\mathcal{D}_e^v(\epsilon)$, where e ranges over the edges ending at v, and we set $\mathcal{G}(\epsilon) = \cup_v \mathcal{G}_v(\epsilon)$. For our purposes, we need to know that for $\mathbf{t}$ close enough to $\mathbf{0}$, in which case the domain $\mathcal{G}(\epsilon)$ is independent of $\mathbf{t}$, the restriction of the $\omega_{j,\mathbf{t}}$'s to $\mathcal{G}(\epsilon)$ depends holomorphically on $\mathbf{t}$, see [20] for more details.

Proposition 10 *The map $\omega \to \Big(\int_{A_e} \omega\Big)_{e\in\mathcal{E}}$ is an isomorphism from $\Omega^1(\Sigma_{\mathbf{t},\mathbf{p}})$ onto the subspace of $\mathbb{C}^{n_e}$ defined by $\{(x_e)_e | \forall v \in \mathcal{V}, \sum\limits_{ter(e)=v} x_e - \sum\limits_{init(e)=v} x_e = 0\}$.*

Proof. We remark that the noted subspace of $\mathbb{C}^{n_e}$ is defined by $n_v - 1$ independent equations and hence its dimension is $n_e - n_v + 1 = n_f + 1 = g$. Furthermore, we recall that for each $\vec{e} = \overrightarrow{vv'}$, the cycle $A_e = C_\epsilon(p_e^v)$ is homotopic to $-C_\epsilon(p_e^{v'})$ and then by applying Cauchy's theorem for $\omega \in \Omega^1(\Sigma_{\mathbf{t},\mathbf{p}})$ in each $\mathcal{G}_v(\epsilon)$ we obtain

$$\sum_{ter(e)=v} \int_{A_e} \omega - \sum_{init(e)=v} \int_{A_e} \omega = 0.$$

Then the image of the noted map is contained in $\{(x_e)_e \in \mathbb{C}^{n_e} | \forall v \in \mathcal{V}, \sum\limits_{ter(e)=v} x_e - \sum\limits_{init(e)=v} x_e = 0\}$. It remains to show that the map is injective. We prove that it is the case for $\mathbf{t} = \mathbf{0}$ and then by continuity the claim will hold for $\mathbf{t} \in \mathcal{N}^*(0)$, which will be sufficient for our purposes. We suppose that $\omega \in \Omega^1(\Sigma_{\mathbf{0},\mathbf{p}})$ with $\displaystyle\int_{A_e} w = 0$ for each $e \in \mathcal{E}$. Then for each $v \in \mathcal{V}$ and for each $e \in \mathcal{E}_v$, the residues of ω at the points $p_e^v \in \widehat{\mathbb{C}_v}$ are all zero, which means that ω is holomorphic in $\widehat{\mathbb{C}_v}$. Therefore ω must be identically zero and the proof is completed. □

3.4.5 The Weierstrass data

We proceed with defining the Weierstrass data for our surface on $\Sigma_{\mathbf{t},\mathbf{p}}$. We require the real line in $\Sigma_{\mathbf{t},\mathbf{p}}$ to be a line of horizontal reflectional symmetry, *i.e.*, we demand that conjugation in $\Sigma_{\mathbf{t},\mathbf{p}}$ correspond to symmetry on our surface with respect to the plane $x_3 = 0$. This encodes into

$$\text{For all } z \in \Sigma_{\mathbf{t},\mathbf{p}},\ Re\int^{\bar{z}} (\phi_1, \phi_2, \phi_3) = Re\int^{z} (\phi_1, \phi_2, -\phi_3).$$

We are driven to define the Weierstrass data as follows :

Prescribe for every edge $e \in \mathcal{E}$,

$$\int_{A_e} \phi_1 = 2\pi i \alpha_e \qquad \int_{A_e} \phi_2 = 2\pi i \beta_e$$

and

$$\int_{A_e} \phi_3 = T,$$

where $\alpha_e,\ \beta_e \in \mathbb{R}$ and $T = 2\pi$.

As Proposition 10 indicates, these equations define the forms ϕ_i given that for all $v \in \mathcal{V}$,

$$\sum_{init(e)=v} \alpha_e - \sum_{ter(e)=v} \alpha_e = 0, \tag{3.4}$$

$$\sum_{init(e)=v} \beta_e - \sum_{ter(e)=v} \beta_e = 0. \tag{3.5}$$

Our choice of the A-periods for the Weierstrass data may be justified by the following

Proposition 11 *Let $\rho : \Sigma_{\mathbf{t},\mathbf{p}} \to \Sigma_{\mathbf{t},\mathbf{p}}$ such that $\rho(z) = \bar{z}$. Then*

$$\rho^*\phi_i = \overline{\phi_i}\ (i \in \{1, 2\}),$$
$$\rho^*\phi_3 = -\overline{\phi_3}$$

Proof. We remark that ρ is orientation reversing and consequently for $1 \leq i \leq 3$

$$\begin{aligned}\int_{A_e} \rho^*\phi_i &= \int_{\rho(A_e)} \phi_i \\ &= -\int_{A_e} \phi_i.\end{aligned}$$

Then for $i \in \{1, 2\}$, we obtain

$$\begin{aligned}\int_{A_e} \rho^*\phi_i &= \overline{\int_{A_e} \phi_i} \\ &= \int_{A_e} \overline{\phi_i},\end{aligned}$$

which reads as $\rho^*\phi_i$ and $\overline{\phi_i}$, two anti-holomorphic forms, which have the same A-periods hence they are equal. By a similar argument we prove that $\rho^*\phi_3 = -\overline{\phi_3}$. □

The above proposition entails the fact that the Weierstrass data, defined as above, carries the aspired symmetry.

Remark 19 *For* $\mathbf{t} = \mathbf{0}$*, the forms* ϕ_i *degenerate to regular differentials* ϕ_i^0 *defined on the Riemann surface with nodes* $\Sigma_{\mathbf{0},\mathbf{p}}$*. Restricted to each* $\overline{\mathbb{C}_v}$*, these regular differentials induce meromorphic forms which we denote by* $\phi_{i,v}^0$*.*

If $\mathcal{T}$ *is balanced, for each* $v \in \mathcal{V}$ *we may consider a Karcher Saddle tower having its ends in a one-to-one correspondence with the edges ending at* v *and such that the asymptotic plane to each end is parallel to its corresponding edge and pointing in the same direction. For* $v \in \mathcal{V}$*, the corresponding saddle tower is defined by meromorphic Weierstrass data on* $\widehat{\mathbb{C}_v}$ *whose poles we assume real, up to a Moebius transformation of the sphere.*
We let q_e^v *be the pole of the Weierstrass data of the saddle tower where the end corresponds to* $e \in \mathcal{E}_v$*. We set* $p_e^v = q_e^v$ *and we choose the* α_e*'s and the* β_e*'s so that* $\Phi_v^0 = (\phi_{1,v}^0, \phi_{2,v}^0, \phi_{3,v}^0)$ *is the Weierstrass data of the saddle tower placed at* v*. We denote the corresponding values of* α_e *and* β_e *by* α_e^0 *and* β_e^0*.*

In what follows, we emphasize the fact that the parameters α_e's, β_e's and p_e^v's are free parameters to be adjusted. We let $\alpha = (\alpha_e)_{e\in\mathcal{E}} \in \mathbb{R}^{n_e}$, $\beta = (\beta)_{e\in\mathcal{E}} \in \mathbb{R}^{n_e}$ and $\mathbf{p} = (p_e^v)_{\substack{v\in\mathcal{V}\\ e\in\mathcal{E}_v}}$ and we adapt the point of view that for $\mathbf{t} = \mathbf{0}$, the parameters take the central values $\alpha_0 = (\alpha_e^0)_{e\in\mathcal{E}}$, $\beta_0 = (\beta_e^0)_{e\in\mathcal{E}}$ and $\mathbf{p_0} = (q_e^v)_{\substack{v\in\mathcal{V}\\ e\in\mathcal{E}_v}}$ for which Φ_v^0 is the Weierstrass data of the saddle tower which corresponds to v, as explained in the above remark. We notice that for each $v \in \mathcal{V}$,

$$\phi^0_{1,v} = \sum_{init(e)=v} \frac{\alpha^0_e}{z - q^v_e} - \sum_{ter(e)=v} \frac{\alpha^0_e}{z - q^v_e}$$
$$\phi^0_{2,v} = \sum_{init(e)=v} \frac{\beta^0_e}{z - q^v_e} - \sum_{ter(e)=v} \frac{\beta^0_e}{z - q^v_e}$$
$$\phi^0_{3,v} = -\sum_{init(e)=v} \frac{i}{z - q^v_e} + \sum_{ter(e)=v} \frac{i}{z - q^v_e}.$$

3.4.6 The period problem

The defining equations of the Weierstrass data $\Phi = (\phi_1, \phi_2, \phi_3)$ ensure that the periods

$$Per_{A_e}(\Phi) := Re \int_{A_e} \Phi = (0, 0, T),\ (e \in \mathcal{E}).$$

This amounts to the fact that the A_e's open into curves on the surface joining a point and its vertical translate by T. The geometric picture of the surfaces to be constructed leads us to require that

$$\Big(Per_B(\phi_1), Per_B(\phi_2)\Big) = Re\Big(\int_B \phi_1, \int_B \phi_2) = 0 \quad mod\, \Gamma, \tag{3.6}$$

where B ranges over the cycles $\{B_f\}_f$. We will show that the symmetry of the surfaces and the way we perform the gluing process imply that

$$Re(\int_B \phi_3) = 0 \quad mod\, T, \tag{3.7}$$

where B ranges again over the cycles $\{B_f\}_f$ and $\{B_{T_i}\}_i$.

Proposition 12 *For each $B \in \{B_f\}_f \cup \{B_{T_1},\ B_{T_2}\}$ we have*

$$Re \int_B \phi_3 = 0\ mod\ T.$$

Proof. We prove the proposition for the cycles B_f with $\sigma(f) = -1$. A similar proof holds for the cycles B_f with $\sigma(f) = +1$. We first remark that the equation $\rho^*\phi_3 = -\overline{\phi_3}$ implies that ϕ_3 is purely imaginary on curves lying on the real line. Therefore, as $\mathcal{S}_{init(e)}$ and $\mathcal{C}_e$ are segments of the real line,

$$Re \int_B \phi_3 = \sum_{\{e \in B : \sigma(e) = -1\}} Re \int_{C_-^{\frac{1}{2}}(p_e^{ter(e)})} \phi_3.$$

However,

$$
\begin{aligned}
-\int_{A_e}\phi_3 &= \int_{C_-^{\frac{1}{2}}(p_e^{ter(e)})}\phi_3 - \int_{\rho(C_-^{\frac{1}{2}}(p_e^{ter(e)}))}\phi_3 \\
&= \int_{C_-^{\frac{1}{2}}(p_e^{ter(e)})}\phi_3 - \int_{C_-^{\frac{1}{2}}(p_e^{ter(e)})}\rho^*\phi_3 \\
&= \int_{C_-^{\frac{1}{2}}(p_e^{ter(e)})}\phi_3 + \int_{C_-^{\frac{1}{2}}(p_e^{ter(e)})}\overline{\phi_3},
\end{aligned}
$$

which implies that

$$
Re\int_{C_-^{\frac{1}{2}}(p_e^{ter(e)})}\phi_3 = -\frac{T}{2}
$$

Therefore, each edge with a negative sign contributes $-\frac{T}{2}$ to $Re\displaystyle\int_B\phi_3$. By proposition 7 each cycle B contains an even number of edges with a negative sign which implies that

$$
Re\int_B\phi_3 = 0 \bmod T.
$$

A similar argument could be carried to deal with the cycles B_{T_i}. The conclusion follows since each B_{T_i} contains an even number of semi-circles, as explained in Remark 18. $\square$

We now proceed to estimate the integrals of the forms ϕ_i on the cycles B_f and B_{T_j}. For convenience, we say that $e \in B_f$ if $e \in \partial f$ and $e \in B_{T_i}$ if $e \in \mathcal{B}_i$.

Proposition 13 *For each $B \in \{B_f\}_f \cup \{B_{T_1},\ B_{T_2}\}$ we have*

$$
\begin{aligned}
Re\int_B\phi_1 &= \sum_{e\in B}\alpha_e\ln\frac{|t_e|}{\epsilon^2} + \mathcal{A}_B^1(\alpha,\mathbf{t},\mathbf{p}) \\
Re\int_B\phi_2 &= \sum_{e\in B}\beta_e\ln\frac{|t_e|}{\epsilon^2} + \mathcal{A}_B^2(\beta,\mathbf{t},\mathbf{p}),
\end{aligned}
$$

where $\mathcal{A}_B^1$ and $\mathcal{A}_B^2$ are analytic.

Proof. We adapt the proof of Lemma 1 in [34]. We emphasize the dependence of the ϕ_i's on $\mathbf{t}$ and for simplicity of notation we let $\phi_{\mathbf{t}}$ stand for either of the forms ϕ_i, $1 \le i \le 2$. We remark that for $e = vv'$, the curve $\mathcal{C}_e$ and the curves b_i^e's, introduced in the definition of the cycles B, go through the neck which joins $\widehat{\mathbb{C}}_v$ and $\widehat{\mathbb{C}}_{v'}$, and which degenerates to a node when $t_e \to 0$. We detail the proof for the curves B_f with $\sigma(f) = -1$, knowing that similar computations hold in case $\sigma(f) = +1$. We start with noticing that

$$\begin{aligned}
\int_B \phi_{\mathbf{t}} =& \sum_{e\in B}\int_{b_e}\phi_{\mathbf{t}}\\
=& \sum_{\{e\in B:\sigma(e)=-1\}}\Big(\int_{\mathcal{S}_{init(e)}}\phi_{\mathbf{t}}+\int_{\mathcal{C}_e}\phi_{\mathbf{t}}+\int_{C_-^{\frac{1}{2}}(p_e^{ter(e)})}\phi_{\mathbf{t}}\Big)+\\
& \sum_{\{e\in B:\sigma(e)=+1\}}\Big(\int_{\mathcal{S}_{init(e)}}\phi_{\mathbf{t}}+\int_{\mathcal{C}_e}\phi_{\mathbf{t}}\Big)\\
=& \sum_{e\in B}\Big(\int_{\mathcal{C}_e}\phi_{\mathbf{t}}+\Lambda_e\Big),
\end{aligned}$$

where Λ_e depends holomorphically in the parameters since it consists of integrals over curves in $\mathcal{G}(\epsilon)$. To estimate $\int_{\mathcal{C}_e}\phi_{\mathbf{t}}$ we develop $\phi_{\mathbf{t}}$ in Laurent series in the annulus $U_e^{init(e)}$ around $p_e^{init(e)}$, where for simplicity of notation, we denote the parameter $z_e^{init(e)}$ by u. Then in $U_e^{init(e)}$

$$\phi_{\mathbf{t}} = \sum_{n\in\mathbb{Z}} a_n u^n du,$$

where a_n depends holomorphically on all the parameters and is given by

$$a_n = \frac{1}{2\pi i}\int_{A_e}\frac{\phi_{\mathbf{t}}}{u^{n+1}} = \frac{1}{2\pi i}\int_{|u|=\frac{|t_e|}{\epsilon}}\frac{\phi_{\mathbf{t}}}{u^{n+1}}.$$

Hence,

$$\begin{aligned}
\int_{\mathcal{C}_e}\phi_{\mathbf{t}} =& \sum_{n\in\mathbb{Z}} a_n \int_{-\epsilon}^{\frac{-|t_e|}{\epsilon}} u^n du\\
=& a_{-1}\ln\frac{|t_e|}{\epsilon^2} + \sum_{n\neq -1}(-1)^{n+1}\frac{a_{n+1}}{n+1}\Big(\frac{|t_e|^{n+1}}{\epsilon^{n+1}}-\epsilon^{n+1}\Big).
\end{aligned}$$

Again, a similar argument could be carried to deal with the cycles B_{T_i}, where the curves b_i^e introduced in the definition of B_{T_i} are considered instead of the $\mathcal{C}_e$'s. Therefore, for each $B\in\{B_f\}_f\cup\{B_{T_1},\ B_{T_2}\}$

$$\int_B \phi_{\mathbf{t}} = \sum_{e\in B} Res(\phi_{\mathbf{t}}, p_e^{init(e)})\ln\frac{|t_e|}{\epsilon^2} + \mathcal{H}_e.$$

Taking the real part of the above equation the claim follows immediately with the analytic term equal to $Re(\sum_e \mathcal{H}_e)$ of the corresponding holomorphic terms $\mathcal{H}_e$. □

Remark 20 *For* $i \in \{1,2\}$, $\sum_f \sigma(f)\mathcal{A}^i_{B_f} = 0.$

As we explained above, we wish to adjust the parameters to solve the equations

$$Re(\int_B \phi_i) = 0 \text{ mod } \Gamma,\, i \in \{1,2\}.$$

For this purpose we introduce the change of variable

$$\frac{|t_e|}{\epsilon^2} = e^{-\frac{s_e}{\varepsilon^2}},$$

where the s_e's are to be non zero parameters and ε is a parameter which varies near zero. We let $\mathbf{s} = (s_e)_{e\in\mathcal{E}}$, then the period equations in propostion 13 can be rewritten as follows :

$$-\varepsilon^2 Re\int_B \phi_1 = \sum_{e\in B}\alpha_e s_e - \varepsilon^2\mathcal{A}^1_B(\alpha,\mathbf{s},\mathbf{p},\varepsilon) \tag{3.8}$$

$$-\varepsilon^2 Re\int_B \phi_2 = \sum_{e\in B}\beta_e s_e - \varepsilon^2\mathcal{A}^2_B(\beta,\mathbf{s},\mathbf{p},\varepsilon). \tag{3.9}$$

3.4.7 The conformality equations

The minimal immersion defined by the Weierstrass data will be conformal if the quadratic differential

$$\mathcal{Q} = \phi_1^2 + \phi_2^2 + \phi_3^2$$

defined on $\Sigma_{\mathbf{t},\mathbf{p}}$ is identically zero. Again, we emphasize that $\mathcal{Q}$ depends on the parameters which will be adjusted as to make it vanish. However, to show that this is possible, we consider the meromorphic differential form $\varphi = \dfrac{\mathcal{Q}}{\phi_1}$ on $\Sigma_{\mathbf{t},\mathbf{p}}$ and we choose our parameters so that φ will be identically zero.
The form φ has its poles at the zeros of ϕ_1. When $\mathbf{t} = \mathbf{0}$, Remark 19 implies that ϕ_1^0 admits simple zeros in each of the punctured spheres $\widehat{\mathbb{C}}_v - \{p^v_e\}$ as explained in section 4.1. However, for $\mathbf{t} \in \mathcal{N}^*(\mathbf{0})$, ϕ_1 admits simple zeros in $\Sigma_{\mathbf{t},\mathbf{p}}$ as well. As a meromorphic differential with simple poles on a closed Riemann surface is determined by its A-periods and its poles and their corresponding residues, we wish to adjust our parameters so that the following equations hold :

for each $e \in \mathcal{E}$

$$\int_{A_e} \varphi = 0, \tag{3.10}$$

for each zero ζ of ϕ_1

$$Res(\varphi,\zeta) = 0. \tag{3.11}$$

The integrals (3.10) are a priori complex. However, it is straightforward that

$$\overline{\int_{A_e} \varphi} = \int_{A_e} \rho^*(\varphi) = -\int_{A_e} \varphi,$$

which implies that the integrals (3.10) are purely imaginary and the equations (3.10) then correspond to n_e real equations.

We set for $e \in \mathcal{E}$

$$F_e(\alpha, \beta, \mathbf{p}, \mathbf{s}, \varepsilon) = \frac{1}{2\pi i} \int_{A_e} \varphi$$

and we consider the smooth function $F = (F_e)_e$.

3.5 The gluing process. Existence results.

In what follows, we introduce the following change of variables : for each $e \in \mathcal{E}$ we write $\alpha_e = r_e \cos\theta_e$ and $\beta_e = r_e \sin\theta_e$. The equations (3.10) can be summed up into

$$F(\mathbf{r}, \theta, \mathbf{p}, \mathbf{s}, \varepsilon) = \mathbf{0}. \tag{3.12}$$

For $\varepsilon = 0$ we write $\alpha_e^0 = r_e^0 \cos\theta_e^0$ and $\beta_e^0 = r_e^0 \sin\theta_e^0$.

Proposition 14 *The parameters* $\mathbf{r}$ *can be prescribed values, as functions of* θ, $\mathbf{p}$, $\mathbf{s}$ *and* ε *(which varies in a neighborhood of 0) so that the equations (*3.10*) hold.*

Proof. For $\varepsilon = 0$ the equations (3.10) can be computed explicitly in terms of the different parameters as follows : the holomorphic forms ϕ_i degenerate to the regular differentials on $\Sigma_{\mathbf{0},\mathbf{p}}$ which induce the meromorphic forms $\phi_{1,v}^0$, $\phi_{2,v}^0$ and $\phi_{3,v}^0$ with simple poles at the points p_e^v. Given that the respective residues at $p_e^{init(e)}$ of $\phi_{1,v}^0$,$\phi_{2,v}^0$, and $\phi_{3,v}^0$ are α_e^0, β_e^0 and $-i$, it is not difficult to show that for all $e \in \mathcal{E}$,

$$F_e(\alpha_0, \beta_0, \mathbf{p}_0, \mathbf{s}, 0) = Res(\varphi_0, p_e^{init(e)}) = \frac{(\alpha_e^0)^2 + (\beta_e^0)^2 - 1}{\alpha_e^0}.$$

The equations (3.10) therefore reduce at $\varepsilon = 0$ to

$$F(\mathbf{r}_0, \theta_0, \mathbf{p}_0, \mathbf{s}, 0) = \Big(\frac{(r_e^0)^2 - 1}{r_e^0 \cos\theta_e^0}\Big)_{e\in\mathcal{E}} = \mathbf{0}. \tag{3.13}$$

The equation (3.13) admits a solution for $\mathbf{r_0} = (r_e^0)$ with each $r_e^0 = 1$, given that each $\theta_e^0 \neq 0$. The differential $D_{\mathbf{r}}F(\mathbf{r_0}, \theta_0, \mathbf{p}_0, \mathbf{s}, 0)$ is easily seen to be an automorphism of $\mathbb{R}^{n_e}$. The implicit function theorem applied at $\varepsilon = 0$, θ_0, $\mathbf{p}_0$ and arbitrarily fixed values of the parameters $\mathbf{s}$ implies that there exists $\mathbf{r} = \mathbf{r}(\theta, \mathbf{p}, \mathbf{s}, \varepsilon)$, with $\varepsilon \in I = (-\delta, \delta)$ ($\delta > 0$), θ near θ_0 and $\mathbf{p}$ near $\mathbf{p}_0$ such that (3.12) holds. □

We restrict ourselves to the values of the r_e's given by the above proposition and we are prompted to find values of the θ_e's, p_e^v's and the s_e's so that the other equations involving these parameters hold. We suppose that δ (as given in the proof of Proposition 14) is small enough so that for $\varepsilon \in I$, $|t_e| < \epsilon^2$ for each $e \in \mathcal{E}$.

Remark 21 *We set $z_e = s_e r_e e^{i\theta_e}$ and $\boldsymbol{z} = \boldsymbol{z}(\theta, \mathbf{p}, \mathbf{s}, \varepsilon) = (z_e)_e \in \mathbb{C}^{n_e}$. For $\varepsilon = 0$, we let $z_e^0 = r_e^0 s_e e^{i\theta_e} = s_e e^{i\theta_e}$ and $\mathbf{z}(0) = \mathbf{z}_0 = (z_e^0)_e$. Let $\mathbf{x} = (\theta, \mathbf{s})$, $\mathbf{h} = (h_e)_e$ and $\mathbf{k} = (k_e)_e$ where $h_e(\theta, \mathbf{p}, \mathbf{s}, \mathbf{z}, \varepsilon) = k_e(\mathbf{r}, \theta, \mathbf{p}, \mathbf{s}, \mathbf{z}, \varepsilon) = r_e s_e e^{i\theta_e} - z_e$.*

Proposition 15 *We can solve $\mathbf{h}(\theta, \mathbf{p}, \mathbf{s}, \mathbf{z}, \varepsilon) = \mathbf{0}$ for θ and $\mathbf{s}$ as functions of $\mathbf{z}$, $\mathbf{p}$ and ε.*

Proof. Clearly, $h_e(\theta, \mathbf{p}_0, \mathbf{s}, \mathbf{z}_0, 0) = 0$. Again by the implicit function theorem we can solve the preceding system of equations $h_e = 0$ for θ and $\mathbf{s}$ once $D_{\mathbf{x}}\mathbf{h}(\theta, \mathbf{p}_0, \mathbf{s}, \mathbf{z}_0, 0)$ is non-singular. However,

$$\begin{aligned} D_{\mathbf{x}}h(\theta, \mathbf{p}_0, \mathbf{s}, \mathbf{z}_0, 0) &= D_{\mathbf{r}}\mathbf{k}(\mathbf{r}_0, \theta, \mathbf{p}_0, \mathbf{s}, \mathbf{z}_0, 0) \circ D_{\mathbf{x}}\mathbf{r}(\theta, \mathbf{p}_0, \mathbf{s}, 0) + \\ &\qquad D_{\mathbf{x}}\mathbf{k}(\mathbf{r}_0, \theta\ \ , \mathbf{p}_0, \mathbf{s}, \mathbf{z}_0, 0) \\ &= D_{\mathbf{x}}\mathbf{k}(\mathbf{r}_0, \theta, \mathbf{p}_0, \mathbf{s}, \mathbf{z}_0, 0) \end{aligned}$$

since $\mathbf{r}(\theta, \mathbf{p}, \mathbf{s}, 0) = 1$ as shown by (3.13).
It is straight forward to compute $D_{\mathbf{x}}\mathbf{k}(\mathbf{r}_0, \theta, \mathbf{p}_0, \mathbf{z}, 0)$ and since the s_e's are non-zero we see that it is non-singular. Therefore, the equation $\mathbf{h}(\theta, \mathbf{p}, \mathbf{s}, \mathbf{z}, \varepsilon) = \mathbf{0}$ can be solved for θ and $\mathbf{s}$ as functions of $\mathbf{z}$, $\mathbf{p}$ and ε. □

The above proposition then implies that the necessary conditions (3.4) and (3.5) and the period equations (3.8) and (3.9) we wish to solve can be written as

$$F_{T_i}(\mathbf{z}, \mathbf{p}, \varepsilon) = \sum_{e \in \mathcal{B}_{T_i}} z_e - \varepsilon^2 \mathcal{A}_{\mathcal{B}_i}(\mathbf{z}, \mathbf{p}, \mathbf{t}) = T_i\ (i \in \{1, 2\}) \tag{3.14}$$

$$F_f(\mathbf{z}, \mathbf{p}, \varepsilon) = \sum_{e \in \partial f} z_e - \varepsilon^2 \mathcal{A}_{B_f}(\mathbf{z}, \mathbf{p}, \mathbf{t}) = 0\ (f \in \mathcal{F}) \tag{3.15}$$

$$\mathcal{R}_v(\mathbf{z}, \mathbf{p}, \varepsilon) = \sum_{init(e)=v} \frac{z_e}{|z_e|} - \sum_{ter(e)=v} \frac{z_e}{|z_e|} = 0\ (v \in \mathcal{V}) \tag{3.16}$$

We note that

$$\sum_v \mathcal{R}_v = 0 \text{ and } \sum_{f \in \partial f} \sigma(f) F_f = 0.$$

which implies that we have as many equations as variables. Fix $\delta > 0$ as in Remark 21, let $I = (-\delta, \delta)$ and consider the map

$$G : \mathbb{C}^{n_e} \times I \to \mathbb{C}^{n_e+2}$$

where

$$G(\mathbf{z}, \mathbf{p}, \varepsilon) = \big((\mathcal{R}_v(\mathbf{z}, \varepsilon))_v, (F_f(\mathbf{z}, \mathbf{p}, \varepsilon))_f, (F_{T_i}(\mathbf{z}, \mathbf{p}, \varepsilon))_{1\leq i\leq 2}\big).$$

We remark that

$$G(\mathbf{z}, \mathbf{p}, \varepsilon) \in \mathcal{W}_v \times \mathcal{W}_f \times \mathbb{C}^2 \subset \mathbb{C}^{n_e+2},$$

where $\mathcal{W}_v = \{(z_v)_v \in \mathbb{C}^{n_v} : \sum_v z_v = 0\}$ and $\mathcal{W}_f = \{(z_f)_f \in \mathbb{C}^{n_f} : \sum_f \sigma(f) z_f = 0\}$.

Proposition 16 *If the tiling $\mathcal{T}$ is oriented, balanced and rigid, the parameters α, β and $\mathbf{s}$ can be assigned values, as functions of $\mathbf{p}$ and ε, which solve the period problem for ϕ_1 and ϕ_2, and kill the A-periods of φ. More precisely, there exist values of α, β and $\mathbf{s}$ as functions of ε (which varies near 0) and $\mathbf{p}$ (which varies near $\mathbf{p}_0$), such that for $i, j \in \{1, 2\}$, for each edge e and each face f of $\mathcal{T}$,*

$$Re \int_{B_f} \phi_i = 0, \qquad Re \int_{B_{T_j}} \phi_i = -\varepsilon^{-2} T_j^i$$

and

$$\int_{A_e} \varphi = 0$$

where $T_j = T_j^1 + i T_j^2 \in \mathbb{C}$.

Proof. The equations (3.14), (3.15), (3.16) reduce to the equation $G(\mathbf{z}, \mathbf{p}, \varepsilon) = 0$. We show that this equation admits solutions for $\varepsilon \neq 0$ by applying the implicit function theorem for G at $\varepsilon = 0$.

The equation $G(\mathbf{z}, \mathbf{p}_0, 0) = 0$ corresponds to the following system of equations (3.17), (3.18), (3.19)

$$\phi_{\mathcal{B}_{T_i}}(\mathbf{z}) = \sum_{e \in \mathcal{B}_{T_i}} z_e^0 = T_i \ (i \in \{1, 2\}) \tag{3.17}$$

$$\phi_f(\mathbf{z}) = \sum_{e \in \partial f} z_e^0 = 0 \ (f \in \mathcal{F}) \tag{3.18}$$

$$\mathcal{F}_v(\mathbf{z}) = \sum_{init(e)=v} \frac{z_e^0}{|z_e^0|} - \sum_{ter(e)=v} \frac{z_e^0}{|z_e^0|} = 0 \ (v \in \mathcal{V}). \tag{3.19}$$

This system admits a solution $\mathbf{z_0}$ in $\mathbb{C}^{n_e}$ given by the tiling $\mathcal{T}$ if it is oriented and balanced. In fact, for each edge $\vec{e} = \overrightarrow{vv'}$ of $\mathcal{T}$, the tiling furnishes a $z_e^0 = z_{v'} - z_v$ as explained in section 2. These z_e^0's represent the vectors corresponding to the oriented edges of $\mathcal{T}$ and obviously solve the equations (3.18) corresponding to the faces f and those corresponding to the cycles $\mathcal{B}_i$ (3.17). The equations (3.19) corresponding to the vertices are exactly the conditions for $\mathcal{T}$ to be balanced since for the chosen values of the z_e^0's, the function $\mathcal{R}_v$ is the resultant at v as defined in section 2. We remark that this solution is given by the lengths of the edges of $\mathcal{T}$ for the s_e's and their corresponding angles with the real axis for the θ_e's. If necessary, we rotate $\mathcal{T}$ in the plane so that each $\theta_e^0 \neq 0$.

Therefore, there exists a solution of $G(\mathbf{z}, \mathbf{p}_0, 0) = 0$. The existence of solutions of $G(\mathbf{z}, \mathbf{p}, \varepsilon) = 0$ for $\varepsilon \neq 0$ is assured by the implicit function theorem once $D_{\mathbf{z}}G(\mathbf{z}_0, \mathbf{p}_0, 0)$ is an isomorphism of $\mathbb{C}^{n_e}$ onto $\mathcal{W}_v \times \mathcal{W}_f \times \mathbb{C}^2$. We remark that $X = (X_e)_e \in \mathbb{C}^{n_e}$ is in the kernel of $D_{\mathbf{z}}G(\mathbf{z}_0, \mathbf{p}_0, 0)$ if and only if $X \in \mathcal{W}$ and $D\mathcal{R}_{\mathcal{T}}(0)X = 0$, where $\mathcal{R}_{\mathcal{T}}$ and $\mathcal{W}$ are as defined in section 2. Therefore, $X \in \mathcal{W}_v$ and to ensure a trivial kernel for $D_{\mathbf{z}}G(\mathbf{z}_0, \mathbf{p}_0, 0)$, it suffices to assume the rigidity of the tiling. At this point, the conclusion is that the equation $G(\mathbf{z}, \mathbf{p}, \varepsilon) = 0$ can be solved for $\mathbf{z}$ as a function of $\mathbf{p}$ and ε which vary near $\mathbf{p}_0$ and 0 respectively. However by the preceding proposition, θ and $\mathbf{s}$ are given as functions of $\mathbf{z}$, $\mathbf{p}$ and ε. Therefore, the parameters α and β and $\mathbf{s}$ have been obtained as functions of $\mathbf{p}$ and ε which vary near $\mathbf{p}_0$ and 0 respectively. This completes the proof. □

We next show that the parameters $\mathbf{p}$ can be adjusted so that the equations (3.11) hold.

Proposition 17 *The parameters* $\mathbf{p}$ *can be prescribed values as functions of* ε *on a neighborhood of* 0 *so that the equations* (3.11) *hold.*

Remark 22 *When restricted to the values of the parameters* α, β, $\mathbf{s}$ *and* $\mathbf{p}$, *obtained in the propositions* 14, 16 *and* 17 *as functions of* ε *on a neighborhood of* 0, *the immersion defined by the corresponding Weierstrass data on the corresponding Riemann surface* $\Sigma_{\mathbf{t},\mathbf{p}}$ *into* $\mathbb{R}^3/L_\varepsilon$ *is conformal, where* L_ε *is the lattice of* $\mathbb{R}^3$ *generated* $\varepsilon^2 T_1$, $\varepsilon^2 T_2$ *and* $(0, 0, T)$.

In the remaining part of this section, unless otherwise specified, we fix $v \in \mathcal{V}$ and we denote the points p_e^v by an increasing sequence $(p_i^v)_i$, $1 \leq i \leq d(v)$. We recall that $d(v)$ is even and that there are as many edges whose initial point is v as edges whose terminal point is v. We may assume that the correspondence between the punctures and the edges is such that p_i^v corresponds to an an edge whose initial point is v when i is even, and to an edge whose terminal point is v otherwise.

Let I be an interval around 0 in the domain of definition of the parameters α, β and $\mathbf{s}$ as functions of ε and Ω a neighborhood of $\mathbf{p}_0$ as given by propositions 14 and 16. As we explained above $\phi_{1,v}^0$ admits $d(v) - 2$ simple real zeros in $\widehat{\mathbb{C}}_v$, which we denote by $\zeta_{i,v}^0$, $1 \leq i \leq d(v) - 2$.

Remark 23 *For* $\varepsilon \in I$ *and* $\mathbf{p} \in \Omega$, ϕ_1 *admits* $d(v) - 2$ *simple zeros in* $\widehat{\mathbb{C}}_v$ *as well, which we denote by* $\zeta_{i,v}$, $1 \leq i \leq d(v) - 2$. *We note that by applying Cauchy's theorem* *in* $\widehat{\mathbb{C}}_v$ *we obtain*

$$\sum_{i=1}^{d(v)-2} Res(\varphi, \zeta_{i,v}) = 0 \tag{3.20}$$

which implies that $Res(\varphi, \zeta_{d(v)-2,v}) = 0$ *once we have* $Res(\varphi, \zeta_{i,v}) = 0$ *for* $1 \leq i \leq d(v) - 3$.

Let Ω_v be the projection of Ω on $\widehat{\mathbb{C}}_v$, and consider the mapping

$$K_v : \Omega_v \times I \to \mathbb{C}^{d(v)-3}$$

defined by

$$K_v(\mathbf{p}_v, \varepsilon) = \big(Res(\varphi, \zeta_{1,v}), .., Res(\varphi, \zeta_{d(v)-3,v})\big)$$

with $\mathbf{p}_v = (p_i^v)_i$.

Remark 24 *Remark* 23 *implies that Proposition* 17 *follows readily once we show that for each* $v \in \mathcal{V}$, *we can prescribe* $\mathbf{p}_v$ *as a function of* ε *so that* $K_v(\mathbf{p}_v, \varepsilon) = \mathbf{0}$. *We note that for* $v \in \mathcal{V}$, *the equation* $K_v(\mathbf{p}_v, \varepsilon) = \mathbf{0}$, *which we wish to solve for* $\mathbf{p}_v$, *is equivalent to a system of* $d(v) - 3$ *equations in* $d(v)$ *variables. To surmount this obstacle, we remark that the values eventually prescribed to the points* $\mathbf{p}$ *determine the conformal type of* $\Sigma_{\mathbf{t},\mathbf{p}}$, *which remains unvaried if we compose each* $\mathbf{p}_v$ *with Moebius transformations of* $\widehat{\mathbb{C}}_v$. *Since such a transformation is defined by assigning three points in the sphere their images, we may assume that for each* v, $p_i^v = p_i^0$ *for* $d(v) - 2 \leq i \leq d(v)$, *which leaves the equation* $K_v(\mathbf{p}_v, \varepsilon) = \mathbf{0}$ *with* $d(v) - 3$ *variables.*

Proof of Proposition 17. Since for $\varepsilon = 0$, Φ_v^0 defines a conformal immersion on $\widehat{\mathbb{C}_v} - \{p_i^0\}_{1 \leq i \leq d(v)}$ we obtain that $\mathcal{Q}|_{\mathbf{t}=0} = 0$. Then we have $K_v(\mathbf{p^0}_v, 0) = \mathbf{0}$, where $\mathbf{p^0}_v = (p_i^0)_{1 \leq i \leq d(v)}$. By Remark 24, we may assume the last three coordinates of $\mathbf{p}_v$ fixed, and once $D_{\mathbf{p}} K_v(\mathbf{p^0}_v, 0)$ is an isomorphism of $\mathbb{C}^{d(v)-3}$ onto $\mathbb{C}^{d(v)-3}$, the implicit function theorem ensures the existence of solutions $\mathbf{p}_v = \mathbf{p}_v(\varepsilon) \in \Omega_v$, for the equation $K_v(\mathbf{p}_v, \varepsilon) = \mathbf{0}$ with $\varepsilon \neq 0$. For this purpose we suppose that $\mathbf{p}_v(\lambda)$ is a curve in $\widehat{\mathbb{C}}_v$ which starts at $\mathbf{p}_v^0$ when $\lambda = 0$ with $\frac{d}{d\lambda}|_{\lambda=0} K_v(\mathbf{p}_v(\lambda), 0) = \mathbf{0}$. We suppose that $\mathbf{p}_v$ has its last three coordinates fixed, and we show that $\dot{\mathbf{p}}_v(0) = \mathbf{0}$.

Remark 25 *The hypotheses on* $\mathbf{p}_v(\lambda)$, *namely,* $\mathbf{p}_v(0) = \mathbf{p}_v^0$ *and the fact that* $\frac{d}{d\lambda}|_{\lambda=0} K_v(\mathbf{p}_v(\lambda), 0) = \mathbf{0}$, *suggest the existence of meromorphic differential forms,* $\phi_{1,\lambda}$, $\phi_{2,\lambda}$ *and* $\phi_{3,\lambda}$ *on* $\widehat{\mathbb{C}}_v$ *such that*

$$Res(\phi_{1,\lambda}, p_j(\lambda)) = (-1)^j \alpha_j^0, \qquad Res(\phi_{2,\lambda}, p_j(\lambda)) = (-1)^j \beta_j^0$$

and

$$Res(\phi_{3,\lambda}, p_j(\lambda)) = (-1)^{j+1} i, \ 1 \leq j \leq d(v)$$

and that

$$\frac{d}{d\lambda}|_{\lambda=0} Res(\varphi_\lambda, \zeta_i(\lambda)) = 0 \;\; 1 \leq k \leq d(v) - 3, \tag{3.21}$$

where each $\zeta_k(\lambda)$ *is a simple zero of* $\phi_{1,\lambda}$*. Here* $\varphi_\lambda = \dfrac{\mathcal{Q}_\lambda}{\phi_1}$, $\mathcal{Q}_\lambda = \phi_{1,\lambda}^2 + \phi_{2,\lambda}^2 + \phi_{3,\lambda}^2$ *and for* $\lambda = 0$ *we obtain* ϕ_1^0, ϕ_2^0 *and* ϕ_3^0.

The above remark indicates that one obtains a variation of the saddle towers. A proof of the fact that $\dot{\mathbf{p}}_v(0) = 0$ follows by showing that the variation field is a constant vector in $\mathbb{R}^3$. A proof of this fact is given in the appendix. Therefore for each v, the implicit function theorem gives each of the p_e^v's as a function of ε such that the equations (3.11) hold. □

3.6 Proof of the main result

Let $\mathcal{T}$ be a tiling as described in Theorem 13 and let $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{F}$ denote its corresponding sets of vertices, edges and faces respectively. By propositions 14, 16 and 17, there exist smooth functions of ε, $\alpha = (\alpha_e)_{e\in\mathcal{E}}$, $\beta = (\beta_e)_{e\in\mathcal{E}}$ and $\mathbf{t} = (t_e)_{e\in\mathcal{E}}$ in $\mathbb{R}^{n_e}$ and $\mathbf{p} = (p_e^v)_{\substack{v\in\mathcal{V}\\ e\in\mathcal{E}_v}}$ in $\mathbb{C}^{2n_e}$, defined near 0, such that the holomorphic forms $\Phi_\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon, \phi_3^\varepsilon)$ defined on the Riemann surface $\Sigma_\varepsilon := \Sigma_{\mathbf{t},\mathbf{p}}$ by their periods as follows :

$$\int_{A_e} \Phi_\varepsilon = (2\pi i\alpha_e, 2\pi i\beta_e, T), \;\; e \in \mathcal{E}$$

verify the following :

$$Re \int_{B_f} \Phi_\varepsilon = (0, 0, \tau), \;\; f \in \mathcal{F}$$

and

$$Re \int_{B_{T_j}} \Phi_\varepsilon = (-\varepsilon^{-2} T_j, \tau) \in \mathbb{R}^3, \;\; j \in \{1, 2\}$$

with $\tau = 0 \; mod \; T$ where we assume, without loss of generality, that $T = 2\pi$. Moreover,

$$\mathcal{Q}_\varepsilon = \sum_i \phi_i^{\varepsilon 2} = 0 \text{ on } \Sigma_\varepsilon.$$

We fix a point $p_\circ$ such that for each ε small enough, $p_\circ \in \Sigma_\varepsilon$, and we set

$$X_\varepsilon(p) = Re \int_{p_\circ}^{p} \Phi_\varepsilon, \quad p \in \Sigma_\varepsilon.$$

Then X_ε is a harmonic conformal map of Σ_ε into $\mathbb{R}^3/L_\varepsilon$, where L_ε is as in Remark 22.

Proposition 18 *For $\varepsilon \neq 0$ small enough, X_ε is an immersion.*

Proof. We prove that the metric $ds_\varepsilon^2 = \frac{1}{2}\left|\Phi_\varepsilon\right|^2$, X_ε induces on Σ_ε, is regular. This amounts to proving that

$$\left|\Phi_\varepsilon\right|^2 = \left|\phi_1^\varepsilon\right|^2 + \left|\phi_2^\varepsilon\right|^2 + \left|\phi_2^\varepsilon\right|^2 > 0 \text{ in } \Sigma_\varepsilon.$$

We consider the region $\mathcal{G} = \bigcup_v \mathcal{G}_v$ of Σ_ε, as it is defined in section 4, and we note that $\left|\Phi_0\right|^2 > 0$ for $z \in \mathcal{G}$, since each Φ_0^v is the Weierstrass data of a Karcher saddle tower on $\widehat{\mathbb{C}}_v$. However, over the region $\mathcal{G}$, Φ_ε depends smoothly on ε. Thus for ε small enough, we have $|\Phi_\varepsilon| > 0$ in $\mathcal{G}$. It remains to show that $|\Phi_\varepsilon| > 0$ on the annuli joining the different spheres $\widehat{\mathbb{C}}_v$. For this purpose, we show that ϕ_3^ε has no zeros in the noted region. In each $\widehat{\mathbb{C}}_v$, ϕ_3^0 admits $d(v)$ poles and consequently it admits $d(v) - 2$ zeros in $\mathcal{G}_v$. Hence by continuity, for ε small enough, ϕ_3^ε admits

$$\sum_v (d(v) - 2) = 2n_e - 2n_v$$

zeros in $\mathcal{G}$. Now it suffices to show that ϕ_3^ε admits no more zeros in Σ_ε. Since ϕ_3^ε is holomorphic on Σ_ε then the number of zeros of ϕ_3^ε is two less than twice the genus of Σ_ε, *i.e.* $2(n_f + 1) - 2 = 2n_e - 2n_v$. Therefore, ϕ_3^ε has all its zeros in $\mathcal{G}$ and the metric is regular. □

Therefore, for $\varepsilon \neq 0$ small enough, X_ε immerses Σ_ε minimally in $\mathbb{R}/L_\varepsilon$. When lifted to $\mathbb{R}^3$, the surface defined by X_ε is a triply periodic minimal surface. In fact, we have the following

Proposition 19 *For $\varepsilon \neq 0$ small enough, X_ε is an embedding.*

Proof. We fix $v \in \mathcal{V}$, we assume that $p_\circ \in \widehat{\mathbb{C}}_v - \{p_e^v\}_{e \in \mathcal{E}_v}$ and we let

$$X_0^v(p) = Re \int_{p_\circ}^{p} \Phi_0^v,\ p \in \widehat{\mathbb{C}}_v - \{p_e^v\}_{e \in \mathcal{E}_v}.$$

X_0^v embeds $\widehat{\mathbb{C}}_v - \{p_e^v\}_{e \in \mathcal{E}_v}$ in $\mathbb{R}^3$ as a Karcher saddle tower.
Since $\Phi_\varepsilon \to \Phi_0^v$ on compacts of $\widehat{\mathbb{C}}_v - \{p_e^v\}_{e \in \mathcal{E}_v}$ when $\varepsilon \to 0$, then $X_\varepsilon \to X_0^v$ over $\mathcal{G}_v$. This implies that for $\varepsilon \neq 0$ small enough, X_ε embeds $\mathcal{G}_v$ as X_0^v does. To show that X_ε is an embedding on $\mathcal{G} = \bigcup_v \mathcal{G}_v$, it is therefore enough to show that $X_\varepsilon(\mathcal{G}_v)$ and $X_\varepsilon(\mathcal{G}_{v'})$ are disjoint for $v \neq v'$. We note that for $\varepsilon \neq 0$ small enough, $X_\varepsilon(\mathcal{G}_v)$ can be put inside a vertical cylinder C_v as it is the case for $X_0^v(\mathcal{G}_v)$. Therefore it is enough to show that for $\varepsilon \neq 0$ small enough, C_v and $C_{v'}$ are disjoint for $v \neq v'$. We fix two

distinct vertices of $\mathcal{T}$, and we choose $p \in \mathcal{G}_v$ and $p' \in \mathcal{G}_{v'}$. We write $X_\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon)$, and we consider a path $\mathcal{P}$ of edges joining v and v'. Then for $\varepsilon \neq 0$ small enough, and by computations similar to those in Proposition 13,

$$x_1^\varepsilon(p') - x_1^\varepsilon(p) = Re \int_p^{p'} \phi_1^\varepsilon \approx -\varepsilon^{-2} \sum_{e \in \mathcal{P}} \cos \theta_e^0 s_e^0.$$

If the coefficient of ε^{-2} is equal to zero, we consider the difference for

$$x_2^\varepsilon(p') - x_2^\varepsilon(p) \approx -\varepsilon^{-2} \sum_{e \in \mathcal{P}} \sin \theta_e^0 s_e^0$$

where the coefficient of ε^{-2} cannot be also equal to zero since the path $\mathcal{P}$ is not closed. This shows that C_v and $C_{v'}$ can be put arbitrarily far apart. Therefore, for $\varepsilon \neq 0$ small enough, X_ε is an embedding on $\mathcal{G} = \bigcup_v \mathcal{G}_v$.

Let $\mathcal{A}_e$, $e = vv'$, be the annulus of Σ_ε which joins the domains $\mathcal{G}_v$ and $\mathcal{G}_{v'}$, and have the bounding circles $C_\epsilon(p_e^v)$ and $C_\epsilon(p_e^{v'})$ as explained in section 4. We still need to understand the behavior of X_ε on the annuli $\mathcal{A}_e$ of Σ_ε, for each $e = vv' \in \mathcal{E}$. The geometric picture underlying our construction indicates that $X_\varepsilon(\mathcal{A}_e)$ is a graph and we show that it is indeed the case. For this purpose, fix an edge $e = vv'$ and let g_0 and g_0' denote the stereographic projections of the Gauss maps of the Karcher saddle towers, given by X_0^v and $X_0^{v'}$ respectively. The stereographic projection g_ε of the Gauss map of X_ε is given in terms of the forms ϕ_i^ε by

$$g_\varepsilon = -\frac{\phi_1^\varepsilon + i\phi_2^\varepsilon}{\phi_3^\varepsilon}.$$

By a similar argument as in the proof of Proposition 18, ϕ_3^ε admits its zeros in the region $\mathcal{G} \subset \Sigma_\varepsilon$. Therefore, g_ε is holomorphic in a neighborhood of $\mathcal{A}_e$. Let $\eta = g_0(p_e^v) = g_0'(p_e^{v'})$ and note that since g_ε is holomorphic in a neighborhood of $\mathcal{A}_e$ and we have

$$|g_\varepsilon(p) - \eta| \leq \max_{p \in \partial \mathcal{A}_e} |g_\varepsilon(p) - \eta|.$$

However, g_ε converges to g_0 in a neighborhood of $C_\epsilon(p_e^v)$ and to g_0' in a neighborhood of $C_\epsilon(p_e^{v'})$ and the values of both g_0 and g_0' stay close to η for ϵ small enough. Then for $\varepsilon \neq 0$ small enough, the values of g_ε on $\mathcal{A}_e$ remain close to η. Let P be the plane orthogonal to η at a point in $C_\epsilon(p_e^v)$. We show that X_ε is a graph over a region in P. Let $\pi : \mathcal{A}_e \to P$ be the orthogonal projection of X_ε to P. Since g_ε remains close to η over $\mathcal{A}_e$ then π is a local diffeomorphism. However, $\mathcal{A}_e$ is compact and connected then a standard theorem implies that π is then a covering map of $\mathcal{A}_e$ onto $\pi(\mathcal{A}_e)$. We remark that $X_\varepsilon(\mathcal{A}_e)$ is a graph near the circles bounding $\mathcal{A}_e$, again since g_ε stays close to η over there. This means that the covering map π is one sheeted which implies that $X_\varepsilon(\mathcal{A}_e)$ is a graph over $\pi(\mathcal{A}_e)$. We remark that $X_\varepsilon(\mathcal{A}_e)$ and $X_\varepsilon(\mathcal{A}_{e'})$ are disjoint for $e \neq e'$. $\square$

Given the facts established in propositions 18 and 19 the proof of Theorem 13 is straightforward. In fact, we consider the minimal embeddings $Y_\varepsilon = \varepsilon^2 X_\varepsilon$ defined on Σ_ε into $\mathbb{R}^3/L_\varepsilon$, and we let $\mathcal{M}_\varepsilon$ be the lift to $\mathbb{R}^3$ of $Y_\varepsilon(\Sigma_\varepsilon)$.

Annexe A

On the deformation of the Karcher towers

We provide a proof of a result of independent interest concerning the Karcher saddle towers. Roughly speaking, we prove that, up to conformal re-parameterizations of the the sphere, an infinitesimal deformation by minimal surfaces of a Karcher saddle tower, performed by perturbing the ends while keeping fixed the corresponding limit normals, is a translation.

More precisely, and without loss of generality, we consider the the Riemann sphere $\widehat{\mathbb{C}}_v$, for some $v \in \mathcal{V}$, and the corresponding situation of Remark 25. We consider the following deformation of the Karcher saddle tower corresponding to v :

$$\begin{aligned} X_\lambda(z) =& Re \int^z \Phi_\lambda \\ =& Re \int^z (\phi_{1,\lambda}, \phi_{2,\lambda}, \phi_{3,\lambda}), \quad z \in \widehat{\mathbb{C}}_v - \{p_1, ..., p_{d(v)}\}. \end{aligned}$$

Remark 26 *As a meromorphic differential form on the sphere whose poles are simple is determined by its poles and their corresponding residues, we obtain that*

$$\phi_{1,\lambda} = \sum_{j=1}^{d(v)} \frac{(-1)^j \cos\theta_j^0}{z - p_j(\lambda)},\ \phi_{2,\lambda} = \sum_{j=1}^{d(v)} \frac{(-1)^j \sin\theta_j^0}{z - p_j(\lambda)}$$

and

$$\phi_{3,\lambda} = \sum_{j=1}^{d(v)} \frac{(-1)^{j+1} i}{z - p_j(\lambda)}.$$

Proposition 20 *If H_λ denotes the mean curvature of the immersion X_λ then*

$$\frac{d}{d\lambda}|_{\lambda=0} H_\lambda = 0.$$

We prove the following lemmas before proving the above proposition.

Lemma 12 $\frac{d}{d\lambda}|_{\lambda=0}\varphi_\lambda = 0$ *if and only if* $\frac{d}{d\lambda}|_{\lambda=0}\mathcal{Q}_\lambda = 0$.

Proof. Write $\mathcal{Q}_\lambda = F_\lambda(z)dz^2$ and $\phi_1 = f_\lambda(z)dz$.
Then

$$\frac{d}{d\lambda}|_{\lambda=0}\varphi_\lambda = \frac{1}{f_0(z)}\frac{d}{d\lambda}|_{\lambda=0}F_\lambda(z),$$

and the claim follows immediately. □

Lemma 13 *The equation* $\mathcal{Q}_0 = 0$ *and the equations (3.21) imply that*

$$\frac{d}{d\lambda}|_{\lambda=0}\varphi_\lambda = 0.$$

Proof. We note that φ_λ has its poles only at the ζ_i's, since the residues of φ_λ at $p_i(\lambda)$'s can be shown to vanish as in Proposition 14. Therefore,

$$\varphi_\lambda = \sum_i \frac{r_i(\lambda)}{z-\zeta_i(\lambda)}dz,$$

where $r_i(\lambda) = Res(\varphi_\lambda, \zeta_i)$. We take the derivative with respect to λ at $\lambda = 0$ of this expression and we use the the equations (3.21) and that $\mathcal{Q}_0 = 0$ to prove the claim.□

Proof of Proposition 20. We denote the metric X_λ induces on $\widehat{\mathbb{C}}_v$ by g_λ and we let $g_\lambda = (g_{ij})_{1\leq i,j\leq 2}$ in a local coordinate $z = u_1 + iu_2$. Then the quadratic differential $\mathcal{Q}_\lambda$ is given by

$$\mathcal{Q}_\lambda = (g_{11} - g_{22} - 2ig_{12})dz^2$$

and H is given by the formula

$$H_\lambda = \frac{g_{22}b_{11} + g_{11}b_{22} - 2g_{12}b_{12}}{2det(g_{ij})},$$

where $b_{ij} = \frac{\partial^2 X_\lambda}{\partial u_i \partial u_j}.N_\lambda$. Here, N_λ is the Gauss map of X_λ and $(b_{ij})_{1\leq i,j\leq 2}$ its second fundamental form.

By lemmas 1 and 2 we obtain that

$$\frac{d}{d\lambda}|_{\lambda=0}(g_{11} - g_{22}) = 0 \text{ and } \frac{d}{d\lambda}|_{\lambda=0}g_{12} = 0.$$

We note that $H(0) = 0$ since for $\lambda = 0$ we obtain a Karcher saddle tower, and that each X_λ is harmonic in the sphere deprived of the punctures $p_i(\lambda)$ since X_λ is the real part of a holomorphic quantity over there. Given these facts, we take the derivative of H_λ at $\lambda = 0$ and the claim follows readily. $\square$

The formula for the second variation of area then implies that the function

$$u = \left\langle \frac{d}{d\lambda}|_{\lambda=0} X_\lambda, N \right\rangle,$$

where N is the Gauss map of Karcher saddle tower obtained for $\lambda = 0$ is a Jacobi function.

Lemma 14 *There exists $v \in \mathbb{R}^3$ such that $u = \langle v, N\rangle$.*
Proof. By Theorem 1 in [6] it suffices to show that u is a bounded function on the saddle tower. This follows immediately if u is bounded around the points p_j. We recall that the Gauss map of the saddle tower admits a limit value at each p_j which is $N(p_j) = \pm(-\sin\theta_j^0, \cos\theta_j^0, 0)$ and we notice that $\frac{d}{d\lambda}|_{\lambda=0} X_\lambda = Re(\Lambda(z))$ where

$$\Lambda(z) = -\Big(\sum_{j=1}^{d(v)} \frac{(-1)^j \cos\theta_j^0 \dot{p}_j}{z - p_j}, \sum_{j=1}^{d(v)} \frac{(-1)^j \sin\theta_j^0 \dot{p}_j}{z - p_j}, i\sum_{j=1}^{d(v)} \frac{(-1)^{j+1} \dot{p}_j}{z - p_j}\Big).$$

Then around a point p_j we have

$$u = Re\,\langle \Lambda(z), N(z)\rangle = Re(\Lambda_1 N_1 + \Lambda_2 N_2 + \Lambda_3 N_3)$$

with $N = (N_1, N_2, N_3)$ and $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$.
For z near p_j we have

$$N_i(z) = N_i(p_j) + (z - p_j)\frac{\partial N_i}{\partial z}(\tilde{z}_j) + \overline{(z - p_j)}\frac{\partial N_i}{\partial \overline{z}}(\tilde{z}_j)$$

where $\tilde{z}_j$ is on the segment whose extremities p_j and z. A simple computation then implies that u is indeed bounded around p_j and the proof is completed.$\square$ Therefore, the field $Y_\lambda = X_\lambda - \lambda v$ is a tangential, *i.e.* $\dfrac{dY_\lambda}{d\lambda}|_{\lambda=0}$ is tangent to the surface given by Φ_v^0. This permits us to write in local parameters $z = u_1 + iu_2$

$$\frac{d}{d\lambda}|_{\lambda=0} Y_\lambda = \xi_1 \frac{\partial X_0}{\partial u_1} + \xi_2 \frac{\partial X_0}{\partial u_2}.$$

Let $\xi = \xi_1 + i\xi_2$ and $\Phi_v^0 = F_0(z)dz$, then

$$\frac{\partial X_0}{\partial u_1} = Re(F_0(z)), \frac{\partial X_0}{\partial u_2} = -Im(F_0(z)),$$

and

$$\frac{d}{d\lambda}|_{\lambda=0}Y_\lambda = Re(\xi F_0(z)). \tag{A.1}$$

Lemma 15 *The field* $\dfrac{dY_\lambda}{d\lambda}|_{\lambda=0}$ *is holomorphic.*
Proof. We notice that since X_0 is a conformal

$$\begin{aligned}\frac{1}{2}\frac{d}{d\lambda}|_{\lambda=0}(g_{11}) &= \frac{d}{d\lambda}|_{\lambda=0}\left\langle \frac{\partial Y_\lambda}{\partial u_1}, \frac{\partial Y_\lambda}{\partial u_1}\right\rangle \\ &= \left\langle \frac{\partial}{\partial u_1}\frac{d}{d\lambda}|_{\lambda=0}Y_\lambda, \frac{\partial X_0}{\partial u_1}\right\rangle \\ &= g_{11}^0\frac{\partial \xi_1}{\partial u_1} + \xi_1\left\langle \frac{\partial^2 X_0}{\partial u_1^2}, \frac{\partial X_0}{\partial u_1}\right\rangle - \xi_2\left\langle \frac{\partial^2 X_0}{\partial u_1^2}, \frac{\partial X_0}{\partial u_2}\right\rangle.\end{aligned}$$

Similarly,

$$\frac{1}{2}\frac{d}{d\lambda}|_{\lambda=0}(g_{22}) = g_{22}^0\frac{\partial \xi_2}{\partial u_2} + \xi_2\left\langle \frac{\partial^2 X_0}{\partial u_2^2}, \frac{\partial X_0}{\partial u_2}\right\rangle - \xi_1\left\langle \frac{\partial^2 X_0}{\partial u_2^2}, \frac{\partial X_0}{\partial u_1}\right\rangle.$$

Now the facts that X_0 is harmonic and that $\frac{d}{d\lambda}|_{\lambda=0}(g_{11}-g_{22}) = 0$ imply that

$$\frac{\partial \xi_1}{\partial u_1} = \frac{\partial \xi_2}{\partial u_2}.$$

In a similar fashion, and using the fact that $\frac{d}{d\lambda}|_{\lambda=0}(g_{12}) = 0$, we can show that

$$\frac{\partial \xi_1}{\partial u_2} = -\frac{\partial \xi_2}{\partial u_1}.$$

Therefore, ξ verifies the Cauchy Riemann equations and hence it is holomorphic. This implies that $\dfrac{dY_\lambda}{d\lambda}|_{\lambda=0}$ is holomorphic and the prof is completed. □

Lemma 16 *The field* $\dfrac{dY_\lambda}{d\lambda}|_{\lambda=0}$ *is identically zero on* $\widehat{\mathbb{C}}_v$*.*
Proof. Let z be the standard coordinate on $\mathbb{C}$. We have the explicit expression for $\dfrac{d}{d\lambda}|_{\lambda=0}Y_\lambda$

$$\tfrac{d}{d\lambda}|_{\lambda=0}Y_\lambda = Re(\Lambda(z)) + \ Cte$$

where G is as in the above lemma.
Therefore, (A.1) implies that $\xi F_0(z) - \Lambda(z) = Cte$ and this equation implies that ξ admits limits at each p_j with $\xi(p_j) = -\dot{p}_j(0)$. As explained in Remark 24, the curve $p(\lambda)$ is such that $p_i(\lambda) = p_i^0$ for $d(v)-2 \leq i \leq d(v)$. Next we recall that the number of zeros of a vector field on $\widehat{\mathbb{C}}_v$ minus the number of it's poles must be

equal to two. However, $\frac{dY_\lambda}{d\lambda}|_{\lambda=0}$ admits three zeros as it is given by $\xi(p_i) = 0$ for $d(v) - 2 \leq i \leq d(v)$. Therefore $\frac{dY_\lambda}{d\lambda}|_{\lambda=0} = 0$ and the proof is completed. $\square$

By the above lemma if z is the standard coordinate in $\mathbb{C}$ and

$$\frac{d}{d\lambda}|_{\lambda=0} Y_\lambda = \xi_1 \frac{\partial X_0}{\partial u_1} + \xi_2 \frac{\partial X_0}{\partial u_2}$$

then $\xi = \xi_1 + i\xi_2 = 0$ and consequently $\xi(p_j) = 0$ of $1 \leq j \leq d(v) - 3$. With the normalization by Moebius transformations, considered in the proof of the above lemma, we obtain that $\dot{\mathbf{p}} = \mathbf{0}$.

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Résumé :

Le cadre de cette thèse est la théorie des surfaces minimales dans deux variétés homogènes, $\mathbb{R}^3$ et $\widetilde{PSL_2}(\mathbb{R})$. Dans $\mathbb{R}^3$, étant donné un pavage $\mathcal{T}$ du plan par des polygones, qui soit invariant par deux translations indépendantes, on construit une famille de surfaces minimales plongées et triplement périodiques qui désingularise $\mathcal{T} \times \mathbb{R}$. Dans cette perspective, et inspiré par le travail de Martin Traizet, nous ouvrons les *nodes* d'une surface de Riemann singulière dans le but de coller ensemble des *Karcher saddle towers*, chacune placée sur un sommet avec ses bouts au long des arrêtes qui se terminent sur ce sommet même. Dans une seconde partie, nous étudions les graphes minimaux dans $\widetilde{PSL_2}(\mathbb{R})$ et nous fournissons des exemples de surfaces invariantes. Nous obtenons des estimées du gradient pour les solutions de l'équation des surfaces minimales dans l'espace en considération et on étudie le comportement des suites monotones de solutions. Nous concluons par prolonger à $\widetilde{PSL_2}(\mathbb{R})$ un théorème de Jenkins et Serrin, qui donnent une condition nécessaire et suffisante pour la solvabilité du problème du Dirichlet de l'équation des surfaces minimales dans $\mathbb{R}^3$, avec des données infinies sur le bord d'un domaine convexe et borné.

Mots clés :

Variétés Homogènes simplement connexes de dimensions trois, Fibrations Riemannienne, Sections minimales, Surfaces minimales invariantes dans $\widetilde{PSL_2}(\mathbb{R})$, Théorème de type Jenkins-Serrin, Surfaces minimales triplement périodiques, Surfaces de Riemann singuliere, différentielles regulière, Karcher Saddle towers, Pavage rigide du plan.

Abstract :

This doctoral thesis deals with minimal surface theory in two homogeneous manifolds, namely, $\mathbb{R}^3$ and $\widetilde{PSL_2}(\mathbb{R})$. In $\mathbb{R}^3$, given a tiling $\mathcal{T}$ of the plane by straight edge polygons, which is invariant by two independent translations, we construct a family of embedded triply periodic minimal surfaces which desingularizes $\mathcal{T} \times \mathbb{R}$. For this purpose, inspired by the work of Martin Traizet, we open the nodes of singular Riemann surfaces to glue together simply periodic Karcher saddle towers, each placed at a vertex of the tiling in such a way that its wings go along the corresponding edges of the tiling ending at that vertex. On the other hand, in $\widetilde{PSL_2}(\mathbb{R})$ we study minimal graphs and we furnish many invariant examples. We derive gradient estimates for solutions of the minimal surface equation in the underlying space and we study convergence of monotone sequences of solutions. Finally, we extend to $\widetilde{PSL_2}(\mathbb{R})$ a result of Jenkins and Serrin who provide a necessary and sufficient condition for the solvability of the Dirichlet problem of the minimal surface equation in $\mathbb{R}^3$, with infinite data over boundary arcs of a convex bounded region.

Keywords :

Homogeneous simply connected 3-manifolds, Riemannian fibrations, Minimal sections, Invariant minimal surfaces in $\widetilde{PSL_2}(\mathbb{R})$, Jenkins-Serrin type theorem, Triply periodic minimal surfaces, Riemann surfaces with nodes, Regular differentials, Karcher saddle towers, Rigid planar tilings.